CHAPTER 6

Fundamental Principles of Traffic Flow

Traffic flow theory involves the development of mathematical relationships among the primary elements of a traffic stream: flow, density, and speed. These relationships help the traffic engineer in planning, designing, and evaluating the effectiveness of implementing traffic engineering measures on a highway system. Traffic flow theory is used in design to determine adequate lane lengths for storing left-turn vehicles on separate left-turn lanes, the average delay at intersections and freeway ramp merging areas, and changes in the level of freeway performance due to the installation of improved vehicular control devices on ramps. Another important application of traffic flow theory is simulation where mathematical algorithms are used to study the complex interrelationships that exist among the elements of a traffic stream or network and to estimate the effect of changes in traffic flow on factors such as crashes, travel time, air pollution, and gasoline consumption.

Methods ranging from physical to empirical have been used in studies related to the description and quantification of traffic flow. This chapter, however, will introduce only those aspects of traffic flow theory that can be used in the planning, design, and operation of highway systems.

6.1 TRAFFIC FLOW ELEMENTS

Let us first define the elements of traffic flow before discussing the relationships among them. However, before we do that, we will describe the time-space diagram which serves as a useful device for defining the elements of traffic flow.

6.1.1 Time-Space Diagram

The time-space diagram is a graph that describes the relationship between the location of vehicles in a traffic stream and the time as the vehicles progress along the

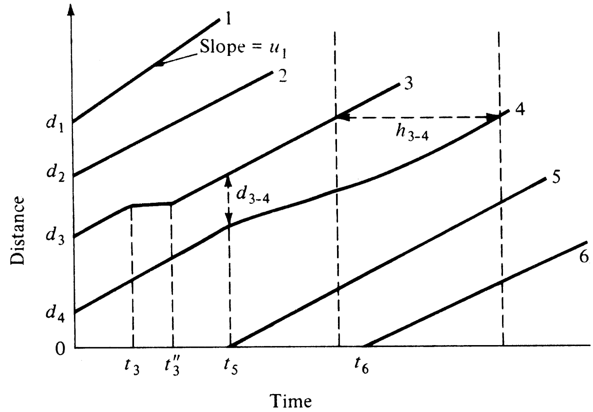


Figure 6.1 Time-Space Diagram

highway. Figure 6.1 shows a time-space diagram for six vehicles with distance plotted on the vertical axis and time on the horizontal axis. At time zero, vehicles 1, 2, 3, and 4 are at respective distances d_1, d_2, d_3, and d_4 from a reference point whereas vehicles 5 and 6 cross the reference point later at times t_5 and t_6, respectively.

6.1.2 Primary Elements of Traffic Flow

The primary elements of traffic flow are flow, density, and speed. Another element, associated with density, is the gap or headway between two vehicles in a traffic stream. The definitions of these elements follow.

Flow

Flow (q) is the equivalent hourly rate at which vehicles pass a point on a highway during a time period less than 1 hour. It can be determined by:

$$q = \frac{n \times 3600}{T} \text{ veh/h} \tag{6.1}$$

where

n = the number of vehicles passing a point in the roadway in T sec
q = the equivalent hourly flow

Density

Density (k), sometimes referred to as *concentration,* is the number of vehicles traveling over a unit length of highway at an instant in time. The unit length is usually 1 mile (mi) thereby making vehicles per mile (veh/mi) the unit of density.

Speed

Speed (u) is the distance traveled by a vehicle during a unit of time. It can be expressed in miles per hour (mi/h), kilometers per hour (km/h), or feet per second (ft/sec).

The speed of a vehicle at any time t is the slope of the time space diagram for that vehicle at time t. Vehicles 1 and 2 in Figure 6.1, for example, are moving at constant speeds because the slopes of the associated graphs are constant. Vehicle 3 moves at a constant speed between time zero and time t_3, then stops for the period t_3 to t_3'' (the slope of graph equals 0), and then accelerates and eventually moves at a constant speed. There are two types of mean speeds: time mean speed and space mean speed.

Time mean speed ($\bar{u}_t$) is the arithmetic mean of the speeds of vehicles passing a point on a highway during an interval of time. The time mean speed is found by:

$$\bar{u}_t = \frac{1}{n} \sum_{i=1}^{n} u_i \tag{6.2}$$

where

n = number of vehicles passing a point on the highway
u_i = speed of the ith vehicle (ft/sec)

Space mean speed ($\bar{u}_s$) is the harmonic mean of the speeds of vehicles passing a point on a highway during an interval of time. It is obtained by dividing the total distance traveled by two or more vehicles on a section of highway by the total time required by these vehicles to travel that distance. This is the speed that is involved in flow-density relationships. The space mean speed is found by

$$\begin{aligned} \bar{u}_s &= \frac{n}{\sum_{i=1}^{n} (1/u_i)} \\ &= \frac{nL}{\sum_{i=1}^{n} t_i} \end{aligned} \tag{6.3}$$

where

$\bar{u}_s$ = space mean speed (ft/sec)
n = number of vehicles
t_i = the time it takes the ith vehicle to travel across a section of highway (sec)
u_i = speed of the ith vehicle (ft/sec)
L = length of section of highway (ft)

The time mean speed is always higher than the space mean speed. The difference between these speeds tends to decrease as the absolute values of speeds increase. It has been shown from field data that the relationship between time mean speed and space mean speed can be given as

$$\bar{u}_t = \bar{u}_s + \frac{\sigma^2}{\bar{u}_s} \tag{6.4}$$

Eq. 6.5 shows a more direct relationship developed by Garber and Sankar using data collected at several sites on freeways. Figure 6.2 also shows a plot of time mean speeds against space mean speeds using the same data.

$$\bar{u}_t = 0.966\bar{u}_s + 3.541 \tag{6.5}$$

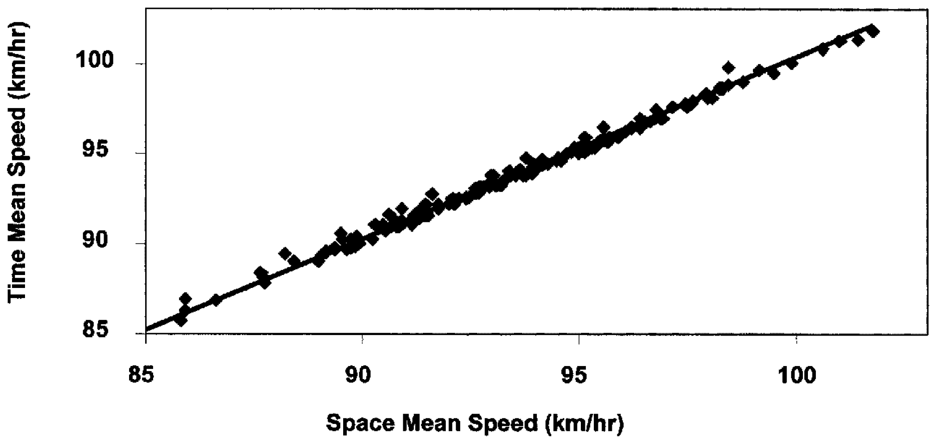


Figure 6.2 Space Mean Speed versus Time Mean Speed

where

$\bar{u}_t$ = time mean speed, km/h
$\bar{u}_s$ = space mean speed, km/h

Time Headway

Time headway (h) is the difference between the time the front of a vehicle arrives at a point on the highway and the time the front of the next vehicle arrives at that same point. Time headway is usually expressed in seconds. For example, in the time space diagram (Figure 6.1), the time headway between vehicles 3 and 4 at d_1 is h_{3-4}.

Space Headways

Space headway (d) is the distance between the front of a vehicle and the front of the following vehicle and is usually expressed in feet. The space headway between vehicles 3 and 4 at time t_5 is d_{3-4} (see Figure 6.1).

Example 6.1 Determining Flow, Density, Time Mean Speed, and Space Mean Speed

Figure 6.3 shows vehicles traveling at constant speeds on a two-lane highway between sections X and Y with their positions and speeds obtained at an instant of time by photography. An observer located at point X observes the four vehicles passing point X during a period of T sec. The velocities of the vehicles are measured as 45, 45, 40, and 30 mi/h, respectively. Calculate the flow, density, time mean speed, and space mean speed.

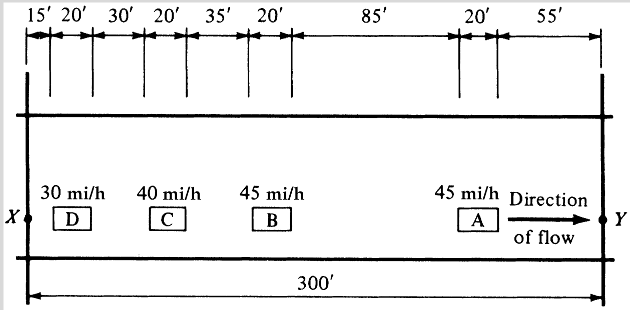


Figure 6.3 Locations and Speeds of Four Vehicles on a Two-Lane Highway at an Instant of Time

Solution: The flow is calculated by

$$\begin{aligned} q &= \frac{n \times 3600}{T} \\ &= \frac{4 \times 3600}{T} = \frac{14{,}400}{T}\ \text{veh/h} \end{aligned} \tag{6.6}$$

With L equal to the distance between X and Y (ft), density is obtained by

$$\begin{aligned} k &= \frac{n}{L} \\ &= \frac{4}{300} \times 5280 = 70.4\ \text{veh/mi} \end{aligned}$$

The time mean speed is found by

$$\begin{aligned} u_t &= \frac{1}{n}\sum_{i=1}^{n} u_i \\ &= \frac{30 + 40 + 45 + 45}{4} = 40\ \text{mi/h} \end{aligned}$$

The space mean speed is found by

$$\begin{aligned} \bar{u}_s &= \frac{n}{\sum_{i=1}^{n}(1/u_i)} \\ &= \frac{Ln}{\sum_{i=1}^{n} t_i} \\ &= \frac{300n}{\sum_{i=1}^{n} t_i} \end{aligned}$$

where t_i is the time it takes the ith vehicle to travel from X to Y at speed u_i, and L (ft) is the distance between X and Y.

$$t_i = \frac{L}{1.47u_i} \text{ sec}$$

$$t_A = \frac{300}{1.47 \times 45} = 4.54 \text{ sec}$$

$$t_B = \frac{300}{1.47 \times 45} = 4.54 \text{ sec}$$

$$t_C = \frac{300}{1.47 \times 40} = 5.10 \text{ sec}$$

$$t_D = \frac{300}{1.47 \times 30} = 6.80 \text{ sec}$$

$$\bar{u}_s = \frac{4 \times 300}{4.54 + 4.54 + 5.10 + 6.80} = 57 \text{ ft/sec}$$
$$= 39.0 \text{ mi/h}$$

6.2 FLOW–DENSITY RELATIONSHIPS

The general equation relating flow, density, and space mean speed is given as

$$\begin{aligned} \text{Flow} &= \text{density} \times \text{space mean speed} \\ q &= k\bar{u}_s \end{aligned} \tag{6.7}$$

Each of the variables in Eq. 6.7 also depends on several other factors including the characteristics of the roadway, characteristics of the vehicle, characteristics of the driver, and environmental factors such as the weather.

Other relationships that exist among the traffic flow variables are given here.

$$\begin{aligned} \text{Space mean speed} &= (\text{flow}) \times (\text{space headway}) \\ \bar{u}_s &= q\bar{d} \end{aligned} \tag{6.8}$$

where

$$\bar{d} = (1/k) = \text{average space headway} \tag{6.9}$$

$$\begin{aligned} \text{Density} &= (\text{flow}) \times (\text{travel time for unit distance}) \\ k &= q\bar{t} \end{aligned} \tag{6.10}$$

where $\bar{t}$ is the average time for unit distance.

$$\begin{aligned} \text{Average space headway} &= (\text{space mean speed}) \times (\text{average time headway}) \\ \bar{d} &= \bar{u}_s\bar{h} \end{aligned} \tag{6.11}$$

$$\begin{aligned}\text{Average time headway} &= (\text{average travel time for unit distance}) \\ &\quad \times (\text{ average space headway}) \\ \bar{h} &= \bar{t}\bar{d} \end{aligned} \tag{6.12}$$

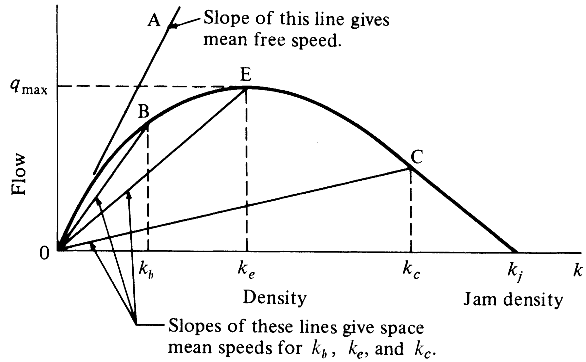
6.2.1 Fundamental Diagram of Traffic Flow

The relationship between the density (veh/mi) and the corresponding flow of traffic on a highway is generally referred to as the fundamental diagram of traffic flow. The following theory has been postulated with respect to the shape of the curve depicting this relationship:

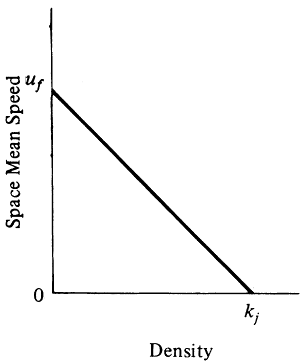
1. When the density on the highway is 0, the flow is also 0 because there are no vehicles on the highway.
2. As the density increases, the flow also increases.
3. However, when the density reaches its maximum, generally referred to as the *jam density* (k_j), the flow must be 0 because vehicles will tend to line up end to end.
4. It follows that as density increases from 0, the flow will also initially increase from 0 to a maximum value. Further continuous increase in density will then result in continuous reduction of the flow, which will eventually be 0 when the density is equal to the jam density. The shape of the curve therefore takes the form in Figure 6.4a on the next page.

Data have been collected that tend to confirm the argument postulated above but there is some controversy regarding the exact shape of the curve. A similar argument can be postulated for the general relationship between the space mean speed and the flow. When the flow is very low, there is little interaction between individual vehicles. Drivers are therefore free to travel at the maximum possible speed. The absolute maximum speed is obtained as the flow tends to 0, and it is known as the *mean free speed* (u_f). The magnitude of the mean free speed depends on the physical characteristics of the highway. Continuous increase in flow will result in a continuous decrease in speed. A point will be reached, however, when the further addition of vehicles will result in the reduction of the actual number of vehicles that pass a point on the highway (that is, reduction of flow). This results in congestion, and eventually both the speed and the flow become 0. Figure 6.4c shows this general relationship. Figure 6.4b shows the direct relationship between speed and density.

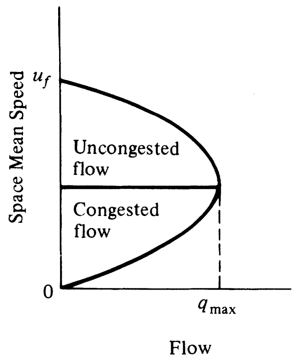
From Eq. 6.7, we know that space mean speed is flow divided by density which makes the slopes of lines 0B, 0C, and 0E in Figure 6.4a represent the space mean speeds at densities k_b, k_c, and k_e, respectively. The slope of line 0A is the speed as the density tends to 0 and little interaction exists between vehicles. The slope of this line is therefore the mean free speed (u_f); it is the maximum speed that can be attained on the highway. The slope of line 0E is the space mean speed for maximum flow. This maximum flow is the capacity of the highway. Thus, it can be seen that it is desirable for highways to operate at densities not greater than that required for maximum flow.



(a) Flow versus density



(b) Space mean speed versus density



(c) Space mean speed versus volume

Figure 6.4 Fundamental Diagrams of Traffic Flow

6.2.2 Mathematical Relationships Describing Traffic Flow

Mathematical relationships describing traffic flow can be classified into two general categories—macroscopic and microscopic—depending on the approach used in the development of these relationships. The macroscopic approach considers flow density relationships whereas the microscopic approach considers spacings between vehicles and speeds of individual vehicles.

Macroscopic Approach

The macroscopic approach considers traffic streams and develops algorithms that relate the flow to the density and space mean speeds. The two most commonly used macroscopic models are the Greenshields and Greenberg models.

Greenshields Model. Greenshields carried out one of the earliest recorded works in which he studied the relationship between speed and density. He hypothesized that a linear relationship existed between speed and density which he expressed as

$$\bar{u}_s = u_f - \frac{u_f}{k_j} k \tag{6.13}$$

Corresponding relationships for flow and density and for flow and speed can be developed. Since $q = \bar{u}_s k$, substituting $q/\bar{u}_s$ for k in Eq. 6.13 gives

$$\bar{u}_s^2 = u_f \bar{u}_s - \frac{u_f}{k_j} q \tag{6.14}$$

Also substituting q/k for $\bar{u}_s$ in Eq. 6.13 gives

$$q = u_f k - \frac{u_f}{k_j} k^2 \tag{6.15}$$

Eqs. 6.14 and 6.15 indicate that if a linear relationship in the form of Eq. 6.13 is assumed for speed and density, then parabolic relationships are obtained between flow and density and between flow and speed. The shape of the curve shown in Figure 6.4a will therefore be a parabola. Also, Eqs. 6.14 and 6.15 can be used to determine the corresponding speed and the corresponding density for maximum flow.

Consider Eq. 6.14:

$$\bar{u}_s^2 = u_f \bar{u}_s - \frac{u_f}{k_j} q$$

Differentiating q with respect to $\bar{u}_s$, we obtain

$$2\bar{u}_s = u_f - \frac{u_f}{k_j} \frac{dq}{du_s}$$

that is

$$\frac{dq}{d\bar{u}_s} = u_f \frac{k_j}{u_f} - 2\bar{u}_s \frac{k_j}{u_f} = k_j - 2\bar{u}_s \frac{k_j}{u_f}$$

For maximum flow,

$$\frac{dq}{d\bar{u}_s} = 0 \qquad k_j = 2\bar{u}_s \frac{k_j}{u_f} \qquad u_o = \frac{u_f}{2} \tag{6.16}$$

Thus, the space mean speed u_o at which the volume is maximum is equal to half the free mean speed.

Consider Eq. 6.15:

$$q = u_f k - \frac{u_f}{k_j} k^2$$

Differentiating q with respect to k, we obtain

$$\frac{dq}{dk} = u_f - 2k \frac{u_f}{k_j}$$

For maximum flow,

$$\frac{dq}{dk} = 0$$

$$u_f = 2k\frac{u_f}{k_j} \tag{6.17}$$

$$\frac{k_j}{2} = k_o$$

Thus, at the maximum flow, the density k_o is half the jam density. The maximum flow for the Greenshields relationship can therefore be obtained from Eqs. 6.7, 6.16, and 6.17, as shown in Eq. 6.18:

$$q_{\max} = \frac{k_j u_f}{4} \tag{6.18}$$

Greenberg Model. Several researchers have used the analogy of fluid flow to develop macroscopic relationships for traffic flow. One of the major contributions using the fluid-flow analogy was developed by Greenberg in the form

$$\bar{u}_s = c\ln\frac{k_j}{k} \tag{6.19}$$

Multiplying each side of Eq. 6.19 by k, we obtain

$$\bar{u}_s k = q = ck\ln\frac{k_j}{k}$$

Differentiating q with respect to k, we obtain

$$\frac{dq}{dk} = c\ln\frac{k_j}{k} - c$$

For maximum flow,

$$\frac{dq}{dk} = 0$$

giving

$$\ln\frac{k_j}{k_o} = 1 \tag{6.20}$$

Substituting 1 for (k_j/k_o) in Eq. 6.19 gives

$$u_o = c$$

Thus, the value of c is the speed at maximum flow.

Model Application

Use of these macroscopic models depends on whether they satisfy the boundary criteria of the fundamental diagram of traffic flow at the region that describes the traffic conditions. For example, the Greenshields model satisfies the boundary conditions when the density k is approaching zero as well as when the density is approaching the jam density k_j. The Greenshields model can therefore be used for light or dense traffic. The Greenberg model, on the other hand, satisfies the boundary conditions when the density is approaching the jam density but it does not satisfy the boundary conditions when k is approaching zero. The Greenberg model is therefore useful only for dense traffic conditions.

Calibration of Macroscopic Traffic Flow Models. The traffic models discussed thus far can be used to determine specific characteristics, such as the speed and density at which maximum flow occurs, and the jam density of a facility. This usually involves collecting appropriate data on the particular facility of interest and fitting the data points obtained to a suitable model. The most common method of approach is *regression analysis.* This is done by minimizing the squares of the differences between the observed and expected values of a dependent variable. When the dependent variable is linearly related to the independent variable, the process is known as *linear regression analysis.* When the relationship is with two or more independent variables, the process is known as *multiple linear regression analysis.*

If a dependent variable y and an independent variable x are related by an estimated regression function, then

$$y = a + bx \tag{6.21}$$

The constants a and b could be determined from Eqs. 6.22 and 6.23. (For development of these equations, see Appendix B.)

$$a = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{b}{n}\sum_{i=1}^{n} x_i = \bar{y} - b\bar{x} \tag{6.22}$$

and

$$b = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2} \tag{6.23}$$

where

n = number of sets of observations
x_i = ith observation for x
y_i = ith observation for y

A measure commonly used to determine the suitability of an estimated regression function is the coefficient of determination (or square of the estimated correlation coefficient) R^2, which is given by

$$R^2 = \frac{\sum_{i=1}^{n} (Y_i - \bar{y})^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2} \tag{6.24}$$

where Y_i is the value of the dependent variable as computed from the regression equations. The closer R^2 is to 1, the better the regression fits.

Example 6.2 Fitting Speed and Density Data to the Greenshields Model

Let us now use the data shown in Table 6.1 (columns 1 and 2) to demonstrate the use of the method of regression analysis in fitting speed and density data to the macroscopic models discussed earlier.

Solution: Let us first consider the Greenshields expression

$$\bar{u}_s = u_f - \frac{u_f}{k_j} k$$

Comparing this expression with our estimated regression function, Eq. 6.21, we see that the speed $\bar{u}_s$ in the Greenshields expression is represented by y in the estimated regression function, the mean free speed u_f is represented by a, and the value of the mean free speed u_f divided by the jam density k_j is represented by $-b$. We therefore obtain

$$\sum y_i = 404.8 \qquad \sum x_i = 892 \qquad \bar{y} = 28.91$$

$$\sum x_i y_i = 20619.8 \qquad \sum x_i^2 = 66{,}628 \qquad \bar{x} = 63.71$$

- Using Eqs. 6.22 and 6.23, we obtain

$$a = 28.91 - 63.71b$$

$$b = \frac{20{,}619.8 - \dfrac{(892)(4048)}{14}}{66{,}628 - \dfrac{(892)^2}{14}} = -0.53$$

or

$$a = 28.91 - 63.71(-0.53) = 62.68$$

Since $a = 62.68$ and $b = -0.53$, then $u_f = 62.68$ mi/h, $u_f/k_j = 0.53$, and so $k_j = 118$ veh/mi, and $\bar{u}_s = 62.68 - 0.53k$.

- Using Eq. 6.24 to determine the value of R^2, we obtain $R^2 = 0.95$.
- Using the above estimated values for u_f and k_j, we can determine the maximum flow from Eq. 6.18 as

$$q_{\max} = \frac{k_j u_f}{4} = \frac{118 \times 62.68}{4}$$
$$= 1849 \text{ veh/h}$$

- Using Eq. 6.16 we also obtain the velocity at which flow is maximum, that is, (62.68/2) = 31.3 mi/h, and Eq. 6.17, the density at which flow is maximum, or (118/2) = 59 veh/h.

Table 6.1 Speed and Density Observations at a Rural Road

(a) Computations for Example 6.2

Speed, u_s (mi/h) y_i	*Density, k (veh/mi) x_i*	$x_i y_i$	x_i^2
53.2	20	1064.0	400
48.1	27	1298.7	729
44.8	35	1568.0	1,225
40.1	44	1764.4	1,936
37.3	52	1939.6	2,704
35.2	58	2041.6	3,364
34.1	60	2046.0	3,600
27.2	64	1740.8	4,096
20.4	70	1428.0	4,900
17.5	75	1312.5	5,625
14.6	82	1197.2	6,724
13.1	90	1179.0	8,100
11.2	100	1120.0	10,000
8.0	115	920.0	13,225
$\Sigma = 404.8$	$\Sigma = 892$	$\Sigma = 20{,}619.8$	$\Sigma = 66{,}628.0$
$\bar{y} = 28.91$	$\bar{x} = 63.71$		

(b) Computations for Example 6.3

Speed, u_s (mi/h) y_i	*Density, k (veh/mi)*	*Ln $k_i x_i$*	$x_i y_i$	x_i^2
53.2	20	2.995732	159.3730	8.974412
48.1	27	3.295837	158.5298	10.86254
44.8	35	3.555348	159.2796	12.64050
40.1	44	3.784190	151.746	14.32009
37.3	52	3.951244	147.3814	15.61233
35.2	58	4.060443	142.9276	16.48720
34.1	60	4.094344	139.6171	16.76365
27.2	64	4.158883	113.1216	17.29631
20.4	70	4.248495	86.66929	18.04971
17.5	75	4.317488	75.55605	18.64071
14.6	82	4.406719	64.33811	19.41917
13.1	90	4.499810	58.94750	20.24828
11.2	100	4.605170	51.57791	21.20759
8.0	115	4.744932	37.95946	22.51438
$\Sigma = 404.8$		$\Sigma = 56.71864$	$\Sigma = 1547.024$	$\Sigma = 233.0369$
$\bar{y} = 28.91$		$\bar{x} = 4.05$		

Example 6.3 Fitting Speed and Density Data to the Greenberg Model

The data in Table 6.1b can also be fitted into the Greenberg model shown in Eq. 6.19:

$$\bar{u}_s = c \ln \frac{k_j}{k}$$

which can be written as

$$\bar{u}_s = c \ln k_j - c \ln k \tag{6.25}$$

Solution: Comparing Eq. 6.25 and the estimated regression function Eq. 6.21, we see that $\bar{u}_s$ in the Greenberg expression is represented by y in the estimated regression function, $c \ln k_j$ is represented by a, c is represented by $-b$, and $\ln k$ is represented by x. Table 6.1b shows values for x_i, $x_i y_i$, *and* x_i^2 (Note that these values are computed to a higher degree of accuracy since they involve logarithmic values.) We therefore obtain

$$\sum y_i = 404.8 \qquad \sum x_i = 56.72 \qquad \bar{y} = 28.91$$

$$\sum x_i y_i = 1547.02 \qquad \sum x_i^2 = 233.04 \qquad \bar{x} = 4.05$$

Using Eqs. 6.22 and 6.23, we obtain

$$a = 28.91 - 4.05b$$

$$b = \frac{1547.02 - \dfrac{(56.72)(404.8)}{14}}{233.04 - \dfrac{56.72^2}{14}} = -28.68$$

or

$$a = 28.91 - 4.05(-28.68) = 145.06$$

Since $a = 145.06$ and $b = -28.68$, the speed for maximum flow is $c = 28.68$ mi/h. Finally, since

$$c \ln k_j = 145.06$$

$$\ln k_j = \frac{145.06}{28.68} = 5.06$$

$$k_j = 157 \text{ veh/mi}$$

then

$$\bar{u}_s = 28.68 \ln \frac{157}{k}$$

Obtaining k_o, the density for maximum flow from Eq. 6.20, we then use Eq. 6.7 to determine the value of the maximum flow.

$$\ln k_j = 1 + \ln k_o$$
$$\ln 157 = 1 + \ln k_o$$
$$5.06 = 1 + \ln k_o$$
$$58.0 = k_o$$
$$q_{\max} = 58.0 \times 28.68 \text{ veh/h}$$
$$q_{\max} = 1663 \text{ veh/h}$$

The R^2 based on the Greenberg expression is 0.9 which indicates that the Greenshields expression is a better fit for the data in Table 6.1. Figure 6.5 shows plots of speed versus density for the two estimated regression functions obtained and also for the actual data points. Figure 6.6 shows similar plots for the volume against speed.

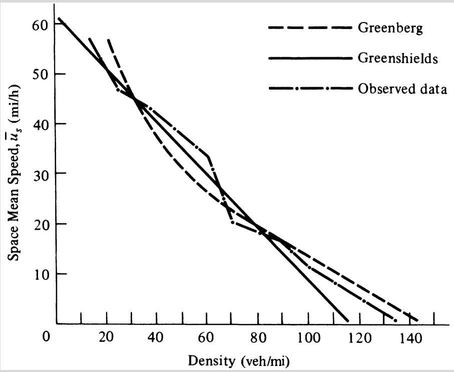


Figure 6.5 Speed versus Density

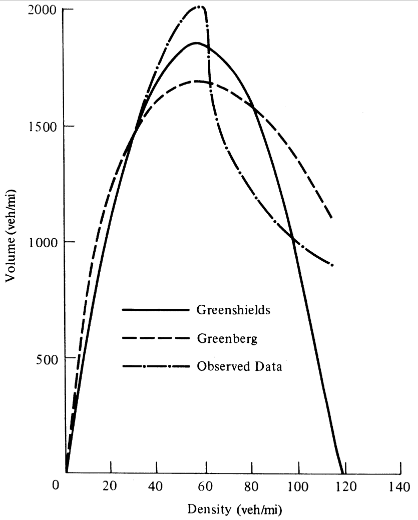


Figure 6.6 Volume versus Density

Software Packages for Linear Regression Analysis

Several software packages are available that can be used to solve the linear regression problem. These include Excel, MiniTab, SAS and SPSS. Appendix C illustrates the use of the Excel spreadsheet to solve Example 6.2.

Microscopic Approach

The microscopic approach, which is sometimes referred to as the car-following theory or the follow-the-leader theory, considers spacings between and speeds of individual vehicles. Consider two consecutive vehicles, A and B, on a single lane of a highway, as shown in Figure 6.7. If the leading vehicle is considered to be the nth vehicle and the following vehicle is considered the $(n + 1)$th vehicle, then the distances of these vehicles from a fixed section at any time t can be taken as x_n and x_{n+1}, respectively.

If the driver of vehicle B maintains an additional separation distance P above the separation distance at rest S such that P is proportional to the speed of vehicle B, then

$$P = \rho \dot{x}_{n+1} \tag{6.26}$$

where

ρ = factor of proportionality with units of time
$\dot{x}_{n+1}$ = speed of the $(n + 1)$th vehicle

We can write

$$x_n - x_{n+1} = \rho \dot{x}_{n+1} + s \tag{6.27}$$

where S is the distance between front bumpers of vehicles at rest.

Differentiating Eq. 6.27 gives

$$\ddot{x}_{n+1} = \frac{1}{\rho}[\dot{x}_n - \dot{x}_{n+1}] \tag{6.28}$$

Eq. 6.28 is the basic equation of microscopic models and it describes the stimulus response of the models. Researchers have shown that a time lag exists for a driver to respond to any stimulus that is induced by the vehicle just ahead and Eq. 6.28 can therefore be written as

$$\ddot{x}_{n+1}(t + T) = \lambda[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{6.29}$$

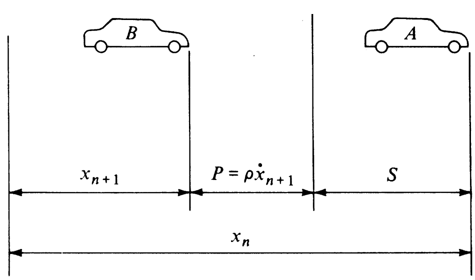


Figure 6.7 Basic Assumptions in Follow-the-Leader Theory

where

T = time lag of response to the stimulus
$\lambda = (1/\rho)$ (sometimes called the sensitivity)

A general expression for λ is given in the form

$$\lambda = a \frac{\dot{x}^m_{n+1}(t + T)}{[x_n(t) - x_{n+1}(t)]^\ell} \tag{6.30}$$

The general expression for the microscopic models can then be written as

$$\ddot{x}_{n+1}(t + T) = a \frac{\dot{x}^m_{n+1}(t + T)}{[x_n(t) - x_{n+1}(t)]^\ell} [\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{6.31}$$

where a, ℓ, and m are constants.

The microscopic model (Eq. 6.31) can be used to determine the velocity, flow, and density of a traffic stream when the traffic stream is moving in a steady state. The direct analytical solution of either Eq. 6.29 or Eq. 6.31 is not easy. It can be shown, however, that the macroscopic models discussed earlier can all be obtained from Eq. 6.31.

For example, if $m = 0$ and $\ell = 1$, the acceleration of the $(n + 1)$th vehicle is given as

$$\ddot{x}_{n+1}(t + T) = a \frac{\dot{x}_n(t) - \dot{x}_{n+1}(t)}{x_n(t) - x_{n+1}(t)}$$

Integrating the above expression, we find that the velocity of the $(n + 1)$th vehicle is

$$\dot{x}_{n+1}(t + T) = a \ln [x_n(t) - x_{n+1}(t)] + C$$

Since we are considering the steady–state condition,

$$\dot{x}_n(t + T) = \dot{x}(t) = u$$

and

$$u = a \ln [x_n - x_{n+1}] + C$$

Also,

$$x_n - x_{n+1} = \text{average space headway} = \frac{1}{k}$$

and

$$u = a \ln \left(\frac{1}{k}\right) + C$$

Using the boundary condition,

$$u = 0$$

when

$$k = k_j$$

and

$$u = 0 = a \ln\left(\frac{1}{k_j}\right) + C$$

$$C = -a \ln\left(\frac{1}{k_j}\right)$$

Substituting for C in the equation for u, we obtain

$$u = a \ln\left(\frac{1}{k}\right) - a \ln\left(\frac{1}{k_j}\right)$$

$$= a \ln\left(\frac{k_j}{k}\right)$$

which is the Greenberg model given in Eq. 6.19. Similarly, if m is allowed to be 0 and $\ell = 2$, we obtain the Greenshields model.

6.3 SHOCK WAVES IN TRAFFIC STREAMS

The fundamental diagram of traffic flow for two adjacent sections of a highway with different capacities (maximum flows) is shown in Figure 6.8. This figure describes the phenomenon of backups and queuing on a highway due to a sudden reduction of the capacity of the highway (known as a *bottleneck condition*). The sudden reduction in capacity could be due to a crash, reduction in the number of lanes, restricted bridge sizes, work zones, a signal turning red, and so forth, creating a situation where the

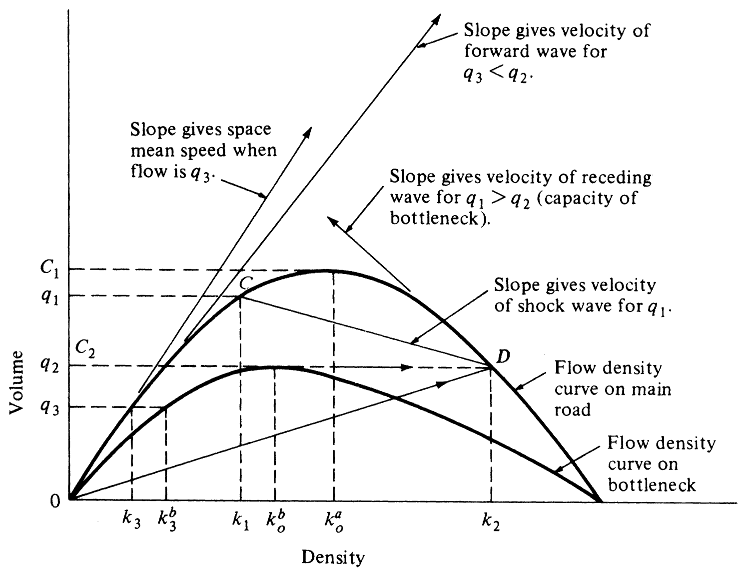


Figure 6.8 Kinematic and Shock Wave Measurements Related to Flow-Density Curve

capacity on the highway suddenly changes from C_1 to a lower value of C_2, with a corresponding change in optimum density from k_o^a to a value of k_o^b.

When such a condition exists and the normal flow and density on the highway are relatively large, the speeds of the vehicles will have to be reduced while passing the bottleneck. The point at which the speed reduction takes place can be approximately noted by the turning on of the brake lights of the vehicles. An observer will see that this point moves upstream as traffic continues to approach the vicinity of the bottleneck indicating an upstream movement of the point at which flow and density change. This phenomenon is usually referred to as a *shock wave* in the traffic stream. The phenomenon also exists when the capacity suddenly increases but in this case, the speeds of the vehicles tend to increase as the vehicles pass the section of the road where the capacity increases.

6.3.1 Types of Shock Waves

Several types of shock waves can be formed, depending on the traffic conditions that lead to their formation. These include ***frontal stationary, backward forming, backward recovery, rear stationary and forward recovery*** shock waves.

Frontal stationary shock waves are formed when the capacity suddenly reduces to zero at an approach or set of lanes having the red indication at a signalized intersection or when a highway is completely closed because of a serious incident. In this case, a frontal stationary shock wave is formed at the stop line of the approach or lanes that have a red signal indication. This type occurs at the location where the capacity is reduced to zero. For example, at a signalized intersection, the red signal indicates that traffic on the approach or set of lanes cannot move across the intersection, which implies that the capacity is temporarily reduced to zero resulting in the formation of a frontal stationary shock wave as shown in Figure 6.9 on the next page.

Backward forming shock waves are formed when the capacity is reduced below the demand flow rate resulting in the formation of a queue upstream of the bottleneck. The shock wave moves upstream with its location at any time indicating the end of the queue at that time. This may occur at the approach of a signalized intersection when the signal indication is red, as shown in Figure 6.9, or at a location of a highway where the number of lanes is reduced.

Backward recovery shock waves are formed when the demand flow rate becomes less than the capacity of the bottleneck or the restriction causing the capacity reduction at the bottleneck is removed. For example, when the signals at an approach or set of lanes on a signalized intersection change from red to green, the traffic flow restriction is removed, and traffic on that approach or set of lanes is free to move across the intersection, causing a backward recovery shock wave as shown in Figure 6.9. The intersection of the backward forming shock wave and the backward recovery shock wave indicates the end of the queue shown as point T in Figure 6.9.

Rear stationary and forward recovery shock waves are formed when demand flow rate upstream of a bottleneck is first higher than the capacity of the bottleneck and then the demand flow rate reduces to the capacity of the bottleneck. For example, consider a four-lane (one direction) highway that leads to a two-lane tunnel in an urban area as shown in Figure 6.10. During the off-peak period when the demand

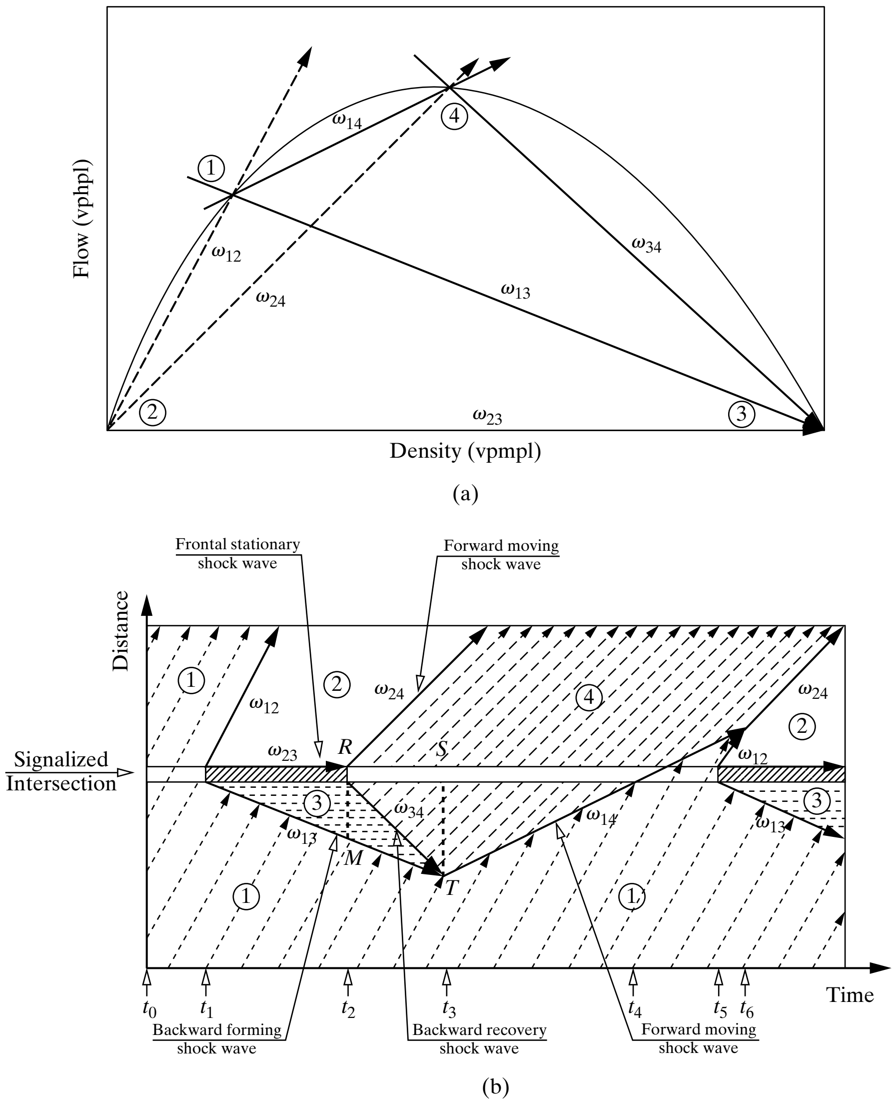


Figure 6.9 Shock Wave at Signalized Intersection

capacity is less than the tunnel capacity, no shock wave is formed. However, when the demand capacity becomes higher than the tunnel capacity during the peak hour, a backward forming shock wave is formed. This shock wave continues to move upstream of the bottleneck as long as the demand flow is higher than the tunnel capacity as shown in Figure 6.10. However. as the end of the peak period approaches, the demand flow rate tends to decrease until it is the same as the tunnel capacity. At this point, a rear stationary shock wave is formed until the demand flow becomes less than the tunnel capacity resulting in the formation of a forward recovery shock wave as shown in Figure 6.10.

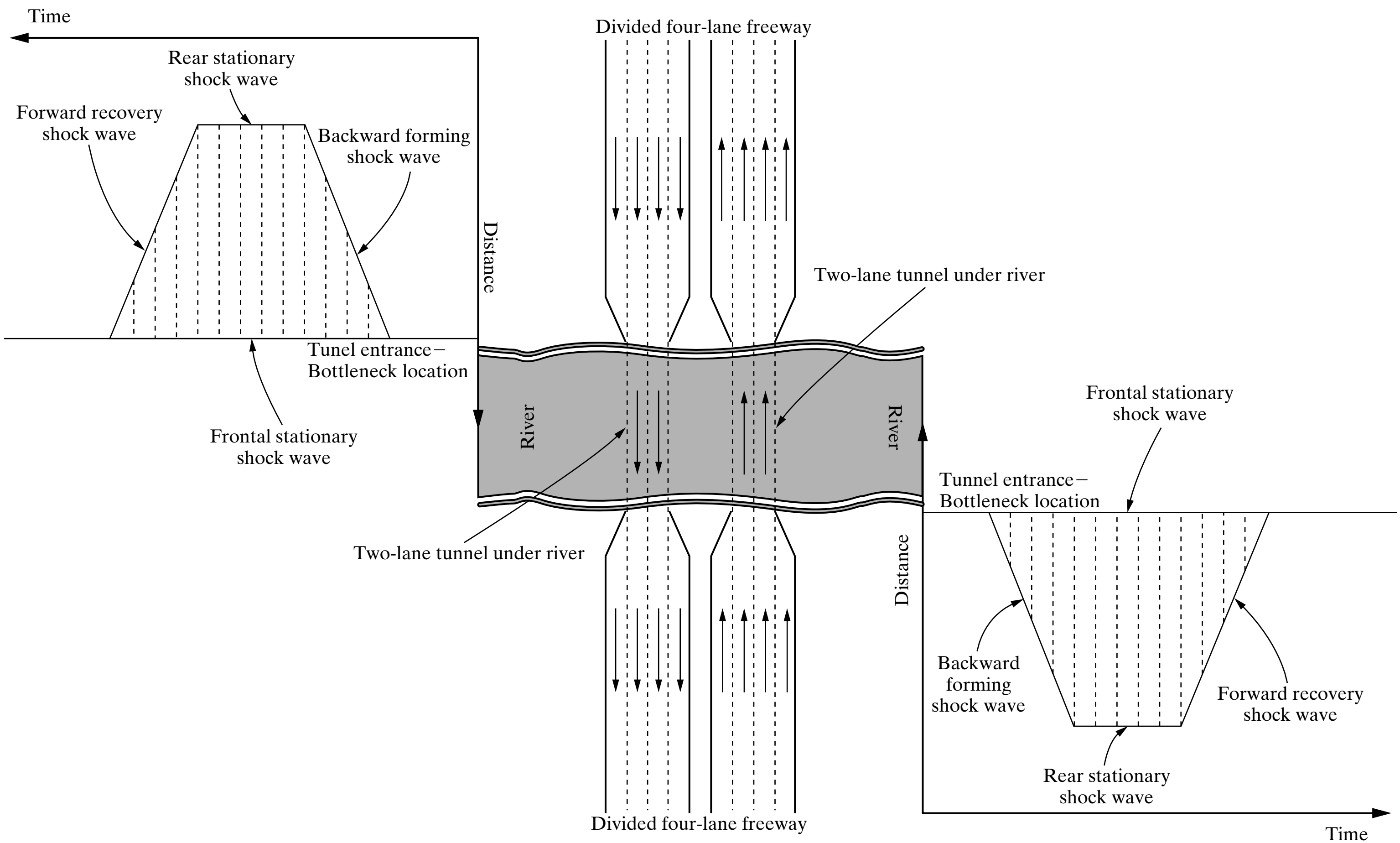


Figure 6.10 Shock Waves Due to a Bottleneck

6.3.2 Velocity of Shock Waves

Let us consider two different densities of traffic, k_1 and k_2, along a straight highway as shown in Figure 6.11 where $k_1 > k_2$. Let us also assume that these densities are separated by the line w representing the shock wave moving at a speed u_w. If the line w moves in the direction of the arrow (that is, in the direction of the traffic flow), u_w is positive.

With u_1 equal to the space mean speed of vehicles in the area with density k_1 (section P), the speed of the vehicle in this area relative to line w is

$$u_{r_1} = (u_1 - u_w)$$

The number of vehicles crossing line w from area P during a time period t is

$$N_1 = u_{r_1} k_1 t$$

Similarly, the speed of vehicles in the area with density k_2 (section Q) relative to line w is

$$u_{r_2} = (u_2 - u_w)$$

and the number of vehicles crossing line w during a time period t is

$$N_2 = u_{r_2} k_2 t$$

Since the net change is zero—that is, $N_1 = N_2$ and $(u_1 - u_w)k_1 = (u_2 - u_w)k_2$—we have

$$u_2 k_2 - u_1 k_1 = u_w (k_2 - k_1) \tag{6.32}$$

If the flow rates in sections P and Q are q_1 and q_2, respectively, then

$$q_1 = k_1 u_1 \qquad q_2 = k_2 u_2$$

Substituting q_1 and q_2 for $k_1 u_1$ and $k_2 u_2$ in Eq. 6.32 gives

$$q_2 - q_1 = u_w (k_2 - k_1)$$

That is

$$u_w = \frac{q_2 - q_1}{k_2 - k_1} \tag{6.33}$$

which is also the slope of the line CD shown in Figure 6.8. This indicates that the velocity of the shock wave created by a sudden change of density from k_1 to k_2 on a traffic stream is the slope of the chord joining the points associated with k_1 and k_2 on the volume density curve for that traffic stream.

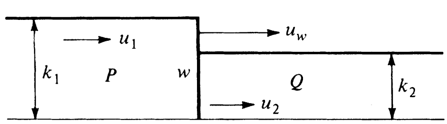


Figure 6.11 Movement of Shock Wave Due to Change in Densities

6.3.3 Shock Waves and Queue Lengths Due to a Red Phase at a Signalized Intersection

Figure 6.9 also shows the traffic conditions that exist at an approach of a signalized intersection when the signal indication is green then changes to red at the end of the green phase (start of the red phase) and changes to green again at the end of the red phase (start of the green phase). When the signal indication is green, the flow is normal as shown in section 1. When the signals change to red at time t_1, two new conditions are formed immediately.

Flow from this approach is stopped creating section 2, immediately downstream of the stop line with a density of zero and flow of zero. At the same time, all vehicles immediately upstream of the stop line are stationary, forming section 3, where the flow is zero and the density is the jam density. This results in the formation of the frontal stationary shock wave with velocity ω_{23} and the backward forming shock wave with velocity ω_{13}.

At the end of the red phase at time t_2 when the signal indication changes to green again, the flow rate at the stop line changes from zero to the saturation flow rate (see Chapter 8 for definition) as shown in section 4. This results in the forward moving shock wave ω_{24}. The queue length at this time—that is at the end of the red phase—is represented by the line RM. Also at this time, the backward recovery shock wave with velocity of ω_{34} is formed that releases the queue as it moves upstream of the stop line. The intersection of the backward forming and backward recovery shock waves at point T and time t_3 indicates the position where the queue is completely dissipated with the maximum queue length being represented by the line ST. The backward forming and backward recovery shock waves also terminate at time t_3 and a new forward moving shock wave with velocity ω_{14} is formed.

When the forward moving shock wave crosses the stop line, at time t_4, the flow changes at the stop line from the saturated flow rate to the original flow rate in section 1 and this continues until time t_5 when the signals change again to red.

Using Eq. 6.33, we can determine expressions for the velocities of the different shock waves and the queue lengths:

$$\text{The shock wave velocity } \omega_{12} = \frac{q_2 - q_1}{k_2 - k_1} = \frac{q_1 - q_2}{k_1 - k_2} = \frac{q_1 - 0}{k_1 - 0} = u_1 \quad (6.34)$$

$$\text{The shock wave velocity } \omega_{13} = \frac{q_1 - q_3}{k_1 - k_3} = \frac{q_1 - 0}{k_1 - k_j} = \frac{q_1}{k_1 - k_j} \quad (6.35)$$

$$\text{The shock wave velocity } \omega_{23} = \frac{q_2 - q_3}{k_2 - k_3} = \frac{0 - 0}{0 - k_j} = \frac{0}{k_j} = 0 \quad (6.36)$$

This confirms that this wave is a stationary wave.

$$\text{The shock wave velocity } \omega_{24} = \frac{q_2 - q_4}{k_2 - k_4} = \frac{0 - q_4}{0 - k_4} = u_4 \quad (6.37)$$

$$\text{The shock wave velocity } \omega_{34} = \frac{q_3 - q_4}{k_3 - k_4} = \frac{0 - q_4}{k_j - k_4} = \frac{-q_4}{k_j - k_4} \quad (6.38)$$

$$\text{The length of the queue at the end of the red signal} = r \times \omega_{13}$$
$$= \frac{rq_1}{k_1 - k_j} \tag{6.39}$$

where r = the length of the red signal indication.

The maximum queue length $\overline{ST}$ can be determined from Figure 6.9 from where it can be seen that $\omega_{34} = \tan\gamma = \dfrac{\overline{ST}}{\overline{RS}}$ which gives $\overline{RS} = \dfrac{\overline{ST}}{\tan\gamma}$. Also, ω_{13} is $\tan\varphi$:

$$\tan\varphi = \frac{\overline{ST}}{r + \overline{RS}}$$

$$\overline{ST} = \tan\varphi\,(r + \overline{RS})$$

$$\overline{ST} = \tan\varphi\left(r + \frac{\overline{ST}}{\tan\gamma}\right)$$

$$r = \frac{\overline{ST}}{\tan\varphi} - \frac{\overline{ST}}{\tan\gamma}$$

$$\overline{ST} = \frac{r}{\dfrac{1}{\tan\varphi} - \dfrac{1}{\tan\gamma}}$$

$$\overline{ST} = \frac{r\tan\varphi\tan\gamma}{\tan\gamma - \tan\varphi}$$

$$\overline{ST} = \frac{r\omega_{13}\omega_{34}}{\omega_{34} - \omega_{13}} \tag{6.40}$$

The additional time $\overline{RS}$ (i.e., $t_3 - t_2$) after the end of the red signal it takes for the maximum queue to be formed can be obtained from the expression $\tan\varphi = \dfrac{\overline{ST}}{r + \overline{RS}}$, which gives

$$\overline{RS} = \frac{\overline{ST}}{\tan\varphi} - r$$

$$= \frac{r\omega_{13}\omega_{34}}{\omega_{13}(\omega_{34} - \omega_{13})} - r$$

$$= \frac{r\omega_{13}}{\omega_{13} - \omega_{34}} \tag{6.41}$$

Example 6.4 Queue Lengths at a Signalized Intersection

The southbound approach of a signalized intersection carries a flow of 1000 veh/h/ln at a velocity of 50 mi/h. The duration of the red signal indication for this approach is 15 sec. If the saturation flow is 2000 veh/h/ln with a density of 75 veh/ln, the jam density is 150 veh/mi, determine the following:

a. The length of the queue at the end of the red phase
b. The maximum queue length
c. The time it takes for the queue to dissipate after the end of the red indication.

Solution:

a. Determine speed of backward forming shock wave ω_{13} when signals turn to red. Use Eq. 6.33.

$$\omega_w = \frac{q_2 - q_1}{k_2 - k_1}$$

$$\omega_{13} = \frac{q_1 - q_3}{k_1 - k_3}$$

$$q_1 = 1000 \text{ veh/h/ln}$$

$$q_3 = 0 \text{ veh/h/ln}$$

$$k_1 = \frac{1000}{50} = 20 \text{ veh/mi (see Eq. 6.7.)}$$

$$\omega_{13} = \frac{1000 - 0}{20 - 150} \text{ mi/h} = -7.69 \text{ mi/h}$$

$$= -7.69 \times 1.47 \text{ ft/sec} = -11.31 \text{ ft/sec}$$

Length of queue at end of red phase = 15 × 11.31 = 169.65 ft

b. Determine speed of backward recovery wave velocity. Use Eq. 6.33.

$$\omega_{34} = \frac{q_3 - q_4}{k_3 - k_4} = \frac{0 - 2000}{150 - 75} = -26.67 \text{ mi/h} = -26.677 \times 1.47 = -39.2 \text{ ft/sec}$$

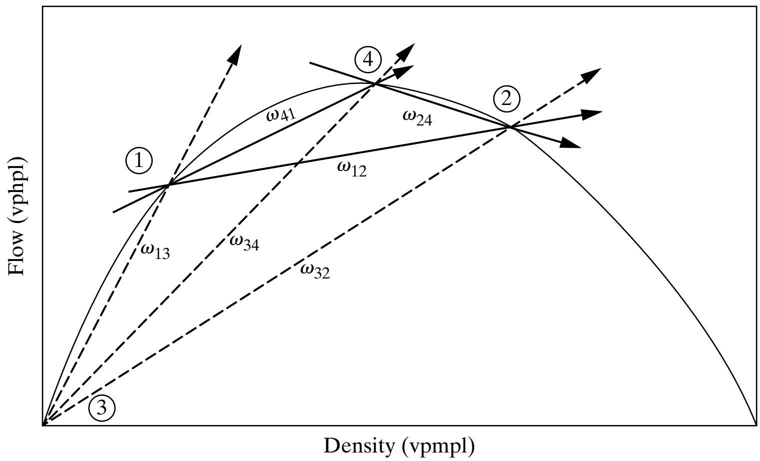
c. Determine the maximum queue length. Use Eq. 6.40.

$$\text{Maximum queue length} = \frac{r\omega_{13}\omega_{34}}{\omega_{34} - \omega_{13}} = \frac{15 \times 11.31 \times 39.2}{39.2 - 11.31} = 238.45 \text{ ft}$$

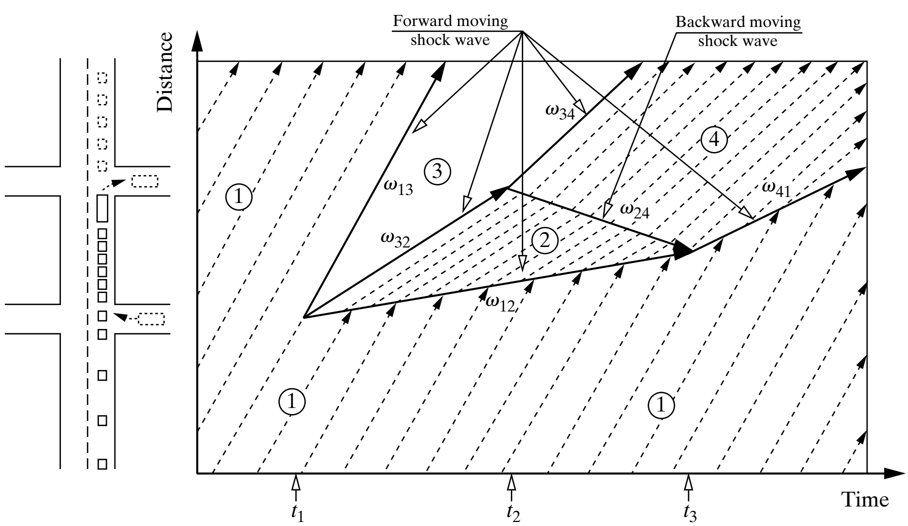
6.3.4 Shock Waves and Queue Lengths Due to Temporary Speed Reduction at a Section of Highway

Let us now consider the situation where the normal speed on a highway is temporarily reduced at a section of a highway where the flow is relatively high but lower than its capacity. For example, consider a truck that enters a two-lane highway at time t_1 and traveling at a much lower speed than the speed of the vehicles driving behind it. The truck travels for some time on the highway and eventually leaves the highway at time t_2. If the traffic condition is such that the vehicles cannot pass the truck, the shock waves that will be formed are shown in Figure 6.12. The traffic conditions prior to the truck entering the highway at time t_1 is depicted as section 1.

At time t_1, vehicles immediately behind the truck will reduce their speed to that of the truck. This results in an increased density immediately behind the truck resulting in traffic condition 2. The moving shock wave with a velocity of ω_{12} is formed. Also, because vehicles ahead of the truck will continue to travel at their original



(a)



(b)

Figure 6.12 Shock Wave Created By Slow Traffic speed, a section on the highway just downstream of the truck will have no vehicles thereby creating traffic condition 3. This also results in the formation of the forward moving shock waves with velocities of ω_{13}, and ω_{32}. At time t_2 when the truck leaves the highway, the flow will be increased to the capacity of the highway with traffic condition 4. This results in the formation of a backward moving shock wave velocity ω_{24} and a forward moving shock wave with velocity ω_{34}. At time t_3, shock waves with velocities ω_{12} and ω_{24} coincide resulting in a new forward moving shock wave with a velocity ω_{41}. It should be noted that the actual traffic conditions 2 and 4 depend on the original traffic condition 1 and the speed of the truck.

Example 6.5 Length of Queue Due to a Speed Reduction

The volume at a section of a two-lane highway is 1500 veh/h in each direction and the density is about 25 veh/mi. A large dump truck loaded with soil from an adjacent construction site joins the traffic stream and travels at a speed of 10 mi/h for a length of 2.5 mi along the upgrade before turning off onto a dump site. Due to the relatively high flow in the opposite direction, it is impossible for any car to pass the truck. Vehicles just behind the truck therefore have to travel at the speed of the truck which results in the formation of a platoon having a density of 100 veh/mi and a flow of 1000 veh/h. Determine how many vehicles will be in the platoon by the time the truck leaves the highway.

Solution: Use Eq. 6.33 to obtain the wave velocity.

$$u_w = \frac{q_2 - q_1}{k_2 - q_1}$$

$$u_w = \frac{1000 - 1500}{100 - 25}$$

$$= -6.7 \text{ mi/h}$$

Knowing that the truck is traveling at 10 mi/h, the speed of the vehicles in the platoon is also 10 mi/h and that the shock wave is moving backward relative to the road at 6.7 mi/h, determine the growth rate of the platoon.

$$10 - (-6.7) = 16.7 \text{ mi/h}$$

Calculate the time spent by the truck on the highway—2.5/10 = 0.25 h—to determine the length of the platoon by the time the truck leaves the highway.

$$0.25 \times 16.7 = 4.2 \text{ mi}$$

Use the density of 100 veh/mi to calculate the number of vehicles in the platoon.

$$100 \times 4.2 = 420 \text{ vehicles}$$

6.3.5 Special Cases of Shock Wave Propagation

The shock wave phenomenon can also be explained by considering a continuous change of flow and density in the traffic stream. If the change in flow and the change in density are very small, we can write

$$(q_2 - q_1) = \Delta q \qquad (k_2 - k_1) = \Delta k$$

The wave velocity can then be written as

$$u_w = \frac{\Delta q}{\Delta k} = \frac{dq}{dk} \tag{6.42}$$

Since q = $k\bar{u}_s$, substituting $k\bar{u}_s$ for q in Eq. 6.42 gives

$$u_w = \frac{d(k\bar{u}_s)}{dk} \tag{6.43}$$

$$= \bar{u}_s + k\frac{d\bar{u}_s}{dk} \tag{6.44}$$

When such a continuous change of volume occurs in a vehicular flow, a phenomenon similar to that of fluid flow exists in which the waves created in the traffic stream transport the continuous changes of flow and density. The speed of these waves is dq/dk and is given by Eq. 6.44.

We have already seen that as density increases, the space mean speed decreases (see Eq. 6.7), giving a negative value for $d\bar{u}_s/dk$. This shows that at any point on the fundamental diagram, the speed of the wave is theoretically less than the space mean speed of the traffic stream. Thus, the wave moves in the opposite direction relative to that of the traffic stream. The actual direction and speed of the wave will depend on the point at which we are on the curve (that is, the flow and density on the highway), and the resultant effect on the traffic downstream will depend on the capacity of the restricted area (bottleneck).

When both the flow and the density of the traffic stream are very low, that is, approaching zero, the flow is much lower than the capacity of the restricted area and there is very little interaction between the vehicles. The differential of $\bar{u}_s$ with respect to k ($d\bar{u}_s/dk$) then tends to zero, and the wave velocity approximately equals the space mean speed. The wave therefore moves forward with respect to the road, and no backups result.

As the flow of the traffic stream increases to a value much higher than zero but still lower than the capacity of the restricted area (say, q_3 in Figure 6.8), the wave velocity is still less than the space mean speed of the traffic stream, and the wave moves forward relative to the road. This results in a reduction in speed and an increase in the density from k_3 to k_3^b as vehicles enter the bottleneck but no backups occur. When the volume on the highway is equal to the capacity of the restricted area (C_2 in Figure 6.8), the speed of the wave is zero and the wave does not move. This results in a much slower speed and a greater increase in the density to k_o^b as the vehicles enter the restricted area. Again, delay occurs but there are no backups.

However, when the flow on the highway is greater than the capacity of the restricted area, not only is the speed of the wave less than the space mean speed of the vehicle stream, but it moves backward relative to the road. As vehicles enter the restricted area, a complex queuing condition arises, resulting in an immediate increase in the density from k_1 to k_2 in the upstream section of the road and a considerable decrease in speed. The movement of the wave toward the upstream section of the traffic stream creates a shock wave in the traffic stream, eventually resulting in backups which gradually moves upstream of the traffic stream.

The expressions developed for the speed of the shock wave, Eqs. 6.33 and 6.44, can be applied to any of the specific models described earlier. For example, the Greenshields model can be written as

$$\bar{u}_{si} = u_f\left(1 - \frac{k_i}{k_j}\right) \qquad \bar{u}_{si} = u_f(1 - \eta_i) \tag{6.45}$$

where $\eta_i = (k_i/k_j)$ (normalized density).

If the Greenshields model fits the flow density relationship for a particular traffic stream, Eq. 6.33 can be used to determine the speed of a shock wave as

$$\begin{aligned}
u_w &= \frac{\left[k_2 u_f\left(1 - \frac{k_2}{k_j}\right)\right] - \left[k_1 u_f\left(1 - \frac{k_1}{k_j}\right)\right]}{k_2 - k_1} \\
&= \frac{k_2 u_f(1 - \eta_2) - k_1 u_f(1 - \eta_1)}{k_2 - k_1} \\
&= \frac{u_f(k_2 - k_1) - k_2 u_f \eta_2 + k_1 u_f \eta_1}{k_2 - k_1} \\
&= \frac{u_f(k_2 - k_1) - \frac{u_f}{k_j}(k_2^2 - k_1^2)}{k_2 - k_1} \\
&= \frac{u_f(k_2 - k_1) - \frac{u_f}{k_j}(k_2 - k_1)(k_2 + k_1)}{(k_2 - k_1)} \\
&= u_f[1 - (\eta_1 + \eta_2)]
\end{aligned}$$

The speed of a shock wave for the Greenshields model is therefore given as

$$u_w = u_f[1 - (\eta_1 + \eta_2)] \tag{6.46}$$

Density Nearly Equal

When there is only a small difference between k_1 and k_2 (that is, $\eta_1 \approx \eta_2$),

$$\begin{aligned}
u_w &= u_f[1 - \eta_1 + \eta_2)] \qquad \text{(neglecting the small change in } \eta_1\text{)} \\
&= u_f[1 - 2\eta_1]
\end{aligned}$$

Stopping Waves

Equation 6.46 can also be used to determine the velocity of the shock wave due to the change from green to red of a signal at an intersection approach if the Greenshields model is applicable. During the green phase, the normalized density is η_1. When the traffic signal changes to red, the traffic at the stop line of the approach comes to a halt which results in a density equal to the jam density. The value of η_2 is then equal to 1.

The speed of the shock wave, which in this case is a stopping wave, can be obtained by

$$u_w = u_f[1 - (\eta_1 + 1)] = -u_f\eta_1 \tag{6.47}$$

Equation 6.47 indicates that in this case the shock wave travels upstream of the traffic with a velocity of $u_f\,\eta_1$. If the length of the red phase is t sec, then the length of the line of cars upstream of the stopline at the end of the red interval is $u_f\,\eta_1 t$.

Starting Waves

At the instant when the signal again changes from red to green, η_1 equals 1. Vehicles will then move forward at a speed of $\bar{u}_{s2}$, resulting in a density of η_2. The speed of the shock wave, which in this case is a starting wave, is obtained by

$$u_w = u_f[1 - (1 + \eta_2)] = -u_f\eta_2 \tag{6.48}$$

Equation 6.45, $\bar{u}_{s2} = u_f(1 - \eta_2)$, gives

$$\eta_2 = 1 - \frac{\bar{u}_{s2}}{u_f}$$

The velocity of the shock wave is then obtained as

$$u_w = -u_f + \bar{u}_{s2}$$

Since the starting velocity $\bar{u}_{s2}$ just after the signal changes to green is usually small, the velocity of the starting shock wave approximately equals $-u_f$.

Example 6.6 Length of Queue Due to a Stopping Shock Wave

Studies have shown that the traffic flow on a single-lane approach to a signalized intersection can be described by the Greenshields model. If the jam density on the approach is 130 veh/mi, determine the velocity of the stopping wave when the approach signal changes to red if the density on the approach is 45 veh/mi and the space mean speed is 40 mi/h. At the end of the red interval, what length of the approach upstream from the stop line will vehicles be affected if the red interval is 35 sec?

Solution:

- Use the Greenshields model.

$$\bar{u}_s = u_f - \frac{u_f}{k_j}k$$

$$40 = u_f - \frac{u_f}{130}45$$

$$5200 = 130u_f - 45u_f$$

$$u_f = 61.2 \text{ mi/h}$$

- Use Eq. 6.47 for a stopping wave.

$$\begin{aligned} u_w &= -u_f\eta_1 \\ &= -61.2 \times \frac{45}{130} \\ &= -21.2 \text{ mi/h} \end{aligned}$$

Since u_w is negative, the wave moves upstream.

- Determine the approach length that will be affected in 35 sec.

$$21.2 \times 1.47 \times 35 = 1090.7 \text{ ft}$$

6.4 GAP AND GAP ACCEPTANCE

Thus far, we have been considering the theory of traffic flow as it relates to the flow of vehicles in a single stream. Another important aspect of traffic flow is the interaction of vehicles as they join, leave, or cross a traffic stream. Examples of these include ramp vehicles merging onto an expressway stream, freeway vehicles leaving the freeway onto frontage roads, and the changing of lanes by vehicles on a multilane highway. The most important factor a driver considers in making any one of these maneuvers is the availability of a gap between two vehicles that, in the driver's judgment, is adequate for him or her to complete the maneuver. The evaluation of available gaps and the decision to carry out a specific maneuver within a particular gap are inherent in the concept of gap acceptance.

Following are the important measures that involve the concept of gap acceptance:

1. **Merging** is the process by which a vehicle in one traffic stream joins another traffic stream moving in the same direction, such as a ramp vehicle joining a freeway stream.
2. **Diverging** is the process by which a vehicle in a traffic stream leaves that traffic stream, such as a vehicle leaving the outside lane of an expressway.
3. **Weaving** is the process by which a vehicle first merges into a stream of traffic, obliquely crosses that stream, and then merges into a second stream moving in the

same direction; for example, the maneuver required for a ramp vehicle to join the far side stream of flow on an expressway.

4. **Gap** is the headway in a major stream, which is evaluated by a vehicle driver in a minor stream who wishes to merge into the major stream. It is expressed either in units of time (time gap) or in units of distance (space gap).
5. **Time lag** is the difference between the time a vehicle that merges into a main traffic stream reaches a point on the highway in the area of merge and the time a vehicle in the main stream reaches the same point.
6. **Space lag** is the difference, at an instant of time, between the distance a merging vehicle is away from a reference point in the area of merge and the distance a vehicle in the main stream is away from the same point.

Figure 6.13 depicts the time-distance relationships for a vehicle at a stop sign waiting to merge and for vehicles on the near lane of the main traffic stream.

A driver who intends to merge must first evaluate the gaps that become available to determine which gap (if any) is large enough to accept the vehicle, in his or her opinion. In accepting that gap, the driver feels that he or she will be able to complete the merging maneuver and safely join the main stream within the length of the gap. This phenomenon is generally referred to as *gap acceptance.* It is of importance when engineers are considering the delay of vehicles on minor roads wishing to join a major-road traffic stream at unsignalized intersections, and also the delay of ramp vehicles wishing to join expressways. It can also be used in timing the release of vehicles at an on ramp of an expressway, such that the probability of the released vehicle finding an acceptable gap in arriving at the freeway shoulder lane is maximum.

To use the phenomenon of gap acceptance in evaluating delays, waiting times, queue lengths, and so forth, at unsignalized intersections and at on-ramps, the average minimum gap length that will be accepted by drivers should be determined first. Several definitions have been given to this "critical" value. Greenshields referred to it as the "acceptable average minimum time gap" and defined it as the gap accepted by 50 percent of the drivers. The concept of "critical gap" was used by Raff, who defined it as the gap for which the number of accepted gaps shorter than it is equal to the

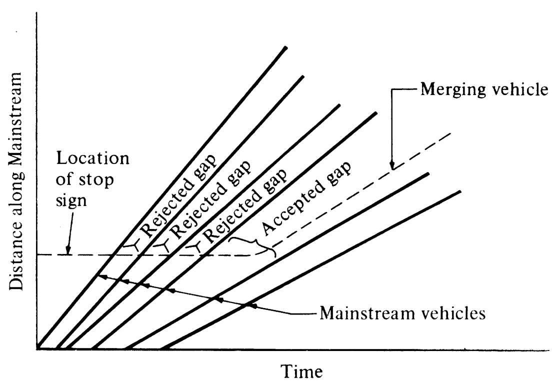


Figure 6.13 Time-Space Diagrams for Vehicles in the Vicinity of a Stop Sign

Table 6.2 Computation of Critical Gap (t_c)

(a) Gaps Accepted and Rejected

1 *Length of Gap* *(t sec)*	*2* *Number of Accepted Gaps* *(less than t sec)*	*3* *Number of Rejected Gaps* *(greater than t sec)*
0.0	0	116
1.0	2	103
2.0	12	66
3.0	$m = 32$	$r = 38$
4.0	$n = 57$	$p = 19$
5.0	84	6
6.0	116	0

(b) Difference in Gaps Accepted and Rejected

1 *Consecutive Gap Lengths (t sec)*	*2* *Change in Number of Accepted Gaps (less than t sec)*	*3* *Change in Number of Rejected Gaps (greater than t sec)*	*4* *Difference Between Columns 2 and 3*
0.0–1.0	2	13	11
1.0–2.0	10	37	27
2.0–3.0	20	28	8
3.0–4.0	25	19	6
4.0–5.0	27	13	14
5.0–6.0	32	6	26

number of rejected gaps longer than it. The data in Table 6.2 are used to demonstrate the determination of the critical gap using Raff's definition. Either a graphical or an algebraic method can be used.

In using the graphical method, two cumulative distribution curves are drawn as shown in Figure 6.14 on page 246. One relates gap lengths t with the number of accepted gaps less than t, and the other relates t with the number of rejected gaps greater than t. The intersection of these two curves gives the value of t for the critical gap.

In using the algebraic method, it is necessary to first identify the gap lengths between where the critical gap lies. This is done by comparing the change in number of accepted gaps less than t sec (column 2 of Table 6.2b) for two consecutive gap lengths, with the change in number of rejected gaps greater than t sec (column 3 of Table 6.2b) for the same two consecutive gap lengths. The critical gap length lies between the two consecutive gap lengths where the difference between the two changes is minimal. Table 6.2b shows the computation and indicates that the critical gap for this case lies between 3 and 4 seconds.

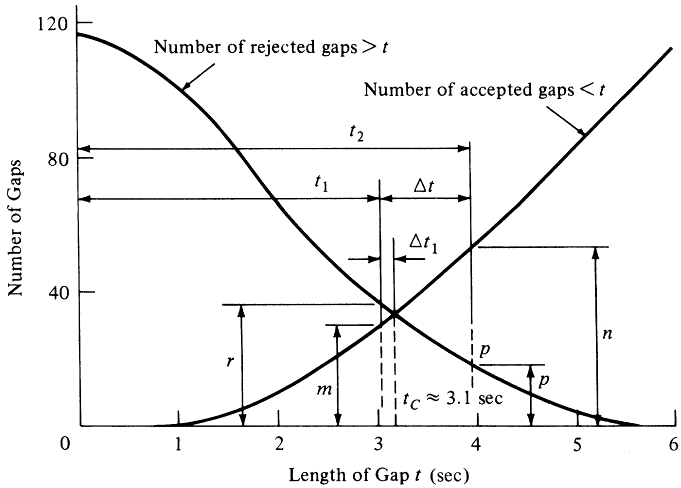


Figure 6.14 Cumulative Distribution Curves for Accepted and Rejected Gaps

For example, in Figure 6.14, with Δt equal to the time increment used for gap analysis, the critical gap lies between t_1 and $t_2 = t_1 + \Delta t$ where

m = number of accepted gaps less than t_1
r = number of rejected gaps greater than t_1
n = number of accepted gaps less than t_2
p = number of rejected gaps greater than t_2

Assuming that the curves are linear between t_1 and t_2, the point of intersection of these two lines represents the critical gap. From Figure 6.14, the critical gap expression can be written as

$$t_c = t_1 + \Delta t_1$$

Using the properties of similar triangles,

$$\frac{\Delta t_1}{r - m} = \frac{\Delta t - \Delta t_1}{n - p}$$

$$\Delta t_1 = \frac{\Delta t(r - m)}{(n - p) + (r - m)}$$

we obtain

$$t_c = t_1 + \frac{\Delta t(r - m)}{(n - p) + (r - m)} \tag{6.49}$$

For the data given in Table 6.2, we thus have

$$t_c = 3 + \frac{1(38 - 32)}{(57 - 19) + (38 - 32)} = 3 + \frac{6}{38 + 6}$$
$$\approx 3.14 \text{ sec}$$

6.4.1 Stochastic Approach to Gap and Gap Acceptance Problems

The use of gap acceptance to determine the delay of vehicles in minor streams wishing to merge onto major streams requires a knowledge of the frequency of arrivals of gaps that are at least equal to the critical gap. This in turn depends on the distribution of arrivals of mainstream vehicles at the area of merge. It is generally accepted that for light to medium traffic flow on a highway, the arrival of vehicles is randomly distributed. It is therefore important that the probabilistic approach to the subject be discussed. It is usually assumed that for light-to-medium traffic the distribution is Poisson, although assumptions of gamma and exponential distributions have also been made.

Assuming that the distribution of mainstream arrival is Poisson, then the probability of x arrivals in any interval of time t sec can be obtained from the expression

$$P(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad (\text{for } x = 0, 1, 2 \ldots, \infty) \tag{6.50}$$

where

$P(x)$ = the probability of x vehicles arriving in time t sec
μ = average number of vehicles arriving in time t

If V represents the total number of vehicles arriving in time T sec, then the average number of vehicles arriving per second is

$$\lambda = \frac{V}{T} \qquad \mu = \lambda t$$

We can therefore write Eq. 6.50 as

$$P(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} \tag{6.51}$$

Now consider a vehicle at an unsignalized intersection or at a ramp waiting to merge into the mainstream flow, arrivals of which can be described by Eq. 6.51. The minor stream vehicle will merge only if there is a gap of t sec equal to or greater than its critical gap. This will occur when no vehicles arrive during a period t sec long. The probability of this is the probability of zero cars arriving (that is, when x in Eq. 6.51 is zero). Substituting zero for x in Eq. 6.51 will therefore give a probability of a gap ($h \geq t$) occurring. Thus,

$$P(0) = P(h \geq t) = e^{-\lambda t} \qquad \text{for } t \geq 0 \tag{6.52}$$

$$P(h < t) = 1 - e^{-\lambda t} \qquad \text{for } t \geq 0 \tag{6.53}$$

Since

$$P(h < t) + P(h \geq t) = 1$$

it can be seen that t can take all values from 0 to ∞, which therefore makes Eqs. 6.52 and 6.53 continuous functions. The probability function described by Eq. 6.52 is known as the *exponential distribution*.

Equation 6.52 can be used to determine the expected number of acceptable gaps that will occur at an unsignalized intersection or at the merging area of an expressway on ramp during a period T, if the Poisson distribution is assumed for the mainstream flow and the volume V is also known. Let us assume that T is equal to 1 hr and that V is the volume in veh/h on the mainstream flow. Since $(V - 1)$ gaps occur between V successive vehicles in a stream of vehicles, then the expected number of gaps greater or equal to t is given as

$$\text{Frequency } (h \geq t) = (V - 1)e^{-\lambda t} \tag{6.54}$$

and the expected number of gaps less than t is given as

$$\text{Frequency } (h < t) = (V - 1)(1 - e^{-\lambda t}) \tag{6.55}$$

Example 6.7 Number of Acceptable Gaps for Vehicles on an Expressway Ramp

The peak hour volume on an expressway at the vicinity of the merging area of an on ramp was determined to be 1800 veh/h. If it is assumed that the arrival of expressway vehicles can be described by a Poisson distribution, and the critical gap for merging vehicles is 3.5 sec, determine the expected number of acceptable gaps for ramp vehicles that will occur on the expressway during the peak hour.

Solution: List the data.

$$V = 1800$$
$$T = 3600 \text{ sec}$$
$$\lambda = (1800/3600) = 0.5 \text{ veh/sec}$$

Calculate the expected number of acceptable gaps in 1 hr using Eq. 6.54.

$$(h \geq t) = (1800 - 1)e^{(-0.5 \times 3.5)} = 1799e^{-1.75} = 312$$

The expected number of occurrences of different gaps t for the previous example have been calculated and are shown in Table 6.3.

Table 6.3 Number of Different Lengths of Gaps Occurring During a Period of 1 hr for $V = 1800$ veh/h and an Assumed Distribution of Poisson for Arrivals

	Probability		*No. of Gaps*	
Gap (t sec)	$P(h \geq t)$	$P(h < t)$	$h \geq t$	$h \leq t$
0	1.0000	0.0000	1799	0
0.5	0.7788	0.2212	1401	398
1.0	0.6065	0.3935	1091	708
1.5	0.4724	0.5276	849	950
2.0	0.3679	0.6321	661	1138
2.5	0.2865	0.7135	515	1284
3.0	0.2231	0.7769	401	1398
3.5	0.1738	0.8262	312	1487
4.0	0.1353	0.8647	243	1556
4.5	0.1054	0.8946	189	1610
5.0	0.0821	0.9179	147	1652

The basic assumption made in this analysis is that the arrival of mainstream vehicles can be described by a Poisson distribution. This assumption is reasonable for light-to-medium traffic but may not be acceptable for conditions of heavy traffic. Analyses of the occurrence of different gap sizes when traffic volume is heavy have shown that the main discrepancies occur at gaps of short lengths (that is, less than 1 second). The reason for this is that although theoretically there are definite probabilities for the occurrence of gaps between 0 and 1 seconds, in reality these gaps very rarely occur, since a driver will tend to keep a safe distance between his or her vehicle and the vehicle immediately in front. One alternative used to deal with this situation is to restrict the range of headways by introducing a minimum gap. Equations 6.54 and 6.55 can then be written as

$$P(h \geq t) = e^{-\lambda(t-\tau)} \qquad \text{(for } t \geq 0\text{)} \tag{6.56}$$

$$P(h < t) = 1 - e^{-\lambda(t-\tau)} \qquad \text{(for } t \leq 0\text{)} \tag{6.57}$$

where τ is the minimum headway introduced.

Example 6.8 Number of Acceptable Gaps with a Restrictive Range, for Vehicles on an Expressway Ramp

Repeat Example 6.7 using a minimum gap in the expressway traffic stream of 1.0 sec and the data:

$$\begin{aligned} V &= 1800 \\ T &= 3600 \\ \lambda &= (1800/3600) = 0.5 \text{ veh/sec} \\ t &= 3.5 \text{ sec} \end{aligned}$$

Solution: Calculate the expected number of acceptable gaps in 1 hr.

$$\begin{aligned} (h \geq t) &= (1800 - 1)e^{-0.5(3.5-1.0)} = 1799e^{-0.5\times 2.5} \\ &= 515 \end{aligned}$$

6.5 INTRODUCTION TO QUEUING THEORY

One of the greatest concerns of traffic engineers is the serious congestion that exists on urban highways, especially during peak hours. This congestion results in the formation of queues on expressway on ramps and off ramps, at signalized and unsignalized intersections, and on arterials, where moving queues may occur. An understanding of the processes that lead to the occurrence of queues and the subsequent delays on highways is essential for the proper analysis of the effects of queuing.

The theory of queuing therefore concerns the use of mathematical algorithms to describe the processes that result in the formation of queues, so that a detailed analysis of the effects of queues can be undertaken. The analysis of queues can be undertaken by assuming either deterministic or stochastic queue characteristics.

6.5.1 Deterministic Analysis of Queues

The deterministic analysis assumes that all the traffic characteristics of the queue are deterministic and demand volumes and capacities are known. There are two common traffic conditions for which the deterministic approach has been used. The first is when an incident occurs on a highway resulting in a significant reduction on the capacity of the highway. This can be described as a varying service rate and constant demand condition. The second is significant increase in demand flow exceeding the capacity of a section of highway which can be described as a varying demand and constant service rate condition.

Varying Service Rate and Constant Demand

Consider a section of three-lane (one-direction) highway with a capacity of c veh/h, i.e., it can serve a maximum volume of c veh/h. (See Chapters 9 and 10 for discussion on capacity.) An incident occurs which resulted in the closure of one lane thereby reducing its capacity to c_R for a period of t hr, which is the time it takes to clear the incident. The demand volume continues to be V veh/h throughout the period of the incident as shown in Figure 6.15a. The demand volume is less than the capacity of the highway section but greater than the reduced capacity. Before the incident, there is no queue as the demand volume is less than the capacity of the highway. However, during the incident the demand volume is higher than the reduced capacity resulting in the formation of a queue as shown in Figure 6.15b. Several important parameters can be determined to describe the effect of this reduction in the highway capacity. These include the maximum queue length, duration of the queue, average queue length, maximum individual delay, time a driver spends in the queue, average queue length while the queue exists, maximum individual delay, and the total delay.

The maximum queue length ($q_{\max}$) is the excess demand rate multiplied by the duration of the incident and is given as

$$q_{\max} = (v - c_R)t \text{ vehicles} \tag{6.58}$$

The time duration of the queue (t_q) is the queue length divided by the difference between the capacity and the demand rate and is given as

$$t_q = \frac{(c - c_R)t}{(c - v)} \text{ hr} \tag{6.59}$$

The average queue length is

$$q_{\text{av}} = \frac{(v - c_R)t}{2} \text{ veh} \tag{6.60}$$

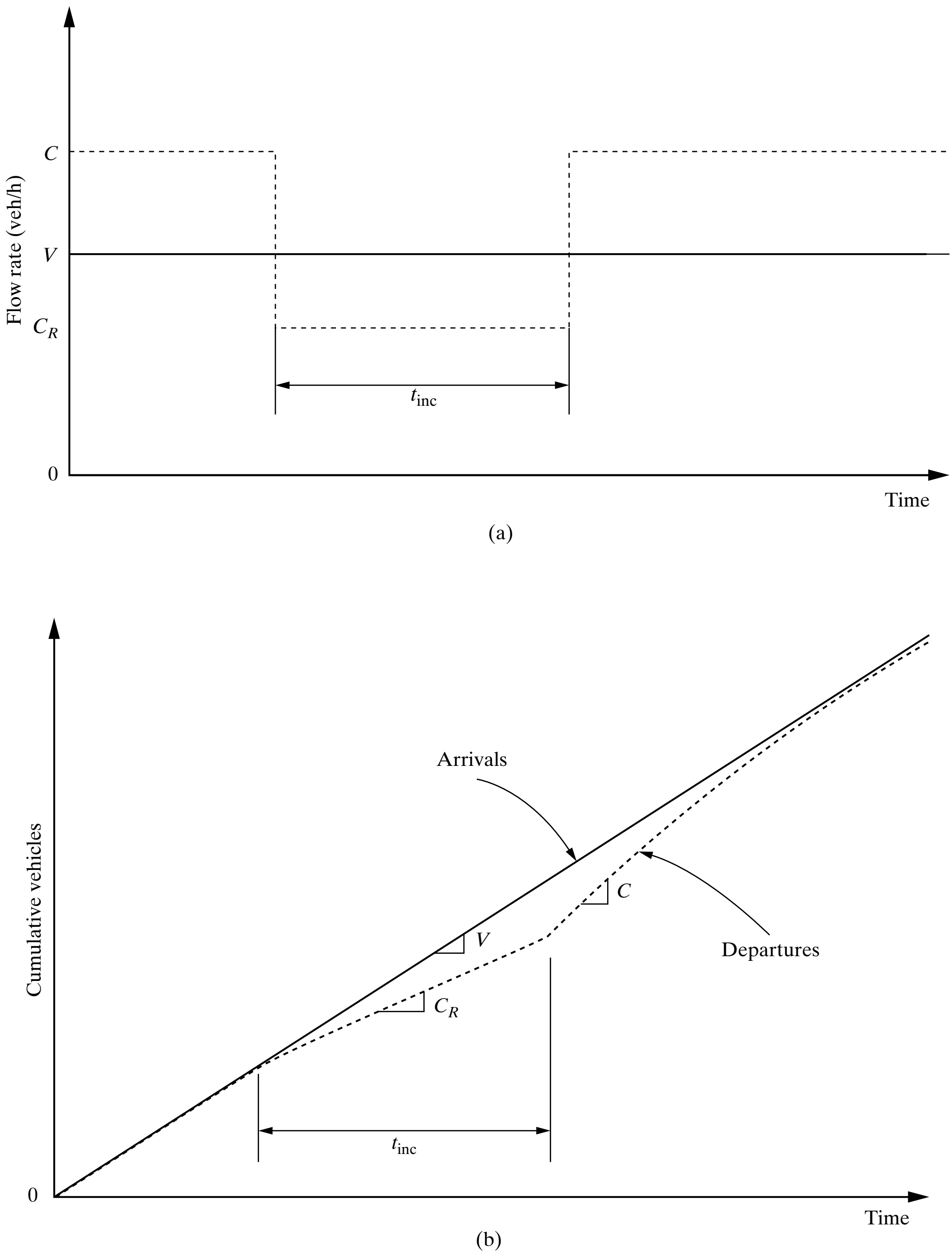


Figure 6.15 Queuing Diagram for Incident Situation

The total delay (d_T) is the time duration of the queue multiplied by the average queue length and is given as

$$d_T = \frac{(v - c_R)t}{2} \frac{(c - c_R)t}{(c - v)} = \frac{t^2(v - c_R)(c - c_R)}{2(c - v)} \text{ hr} \tag{6.61}$$

When using Eqs. 6.57 to 6.61, care should be taken to ensure that the same unit is used for all variables.

Example 6.9 Queue Length and Delay Due to an Incident on a Freeway Using Deterministic Analysis

A three-lane expressway (one direction) is carrying a total volume of 4050 veh/h when an incident occurs resulting in the closure of two lanes. If it takes 90 min to clear the obstruction, determine the following:

a. The maximum queue length that will be formed
b. The total delay
c. The number of vehicles that will be affected by the incident
d. The average individual delay

Assume that the capacity of the highway is 2000 veh/h/ln.

Solution:

- Determine capacity, c, of highway = 3 × 2000 = 6000 veh/h
- Determine reduced capacity, c_R, of highway = 2000 × (3 − 2) = 2000 veh/h
- Duration of incident = 90 min = $1^1/_2$ h

a. Determine maximum queue length. Use Eq. 6.58.

$$q_{\max} = (v - c_R)t \text{ vehicles} = (4050 - 2000) \times 1.5 \text{ veh} = 3075 \text{ veh}$$

b. Determine the total delay—use Eq. 6.61.

$$d_T = \frac{t^2(v - c_R)(c - c_R)}{2(c - v)} = \frac{1.5^2(4050 - 2000)(6000 - 2000)}{2(6000 - 4500)}$$
$$= 6150 \text{ hr}$$

c. Determine the number of vehicles that will be affected by the incident = the demand rate multiplied by the duration of the incident = 4050 × 1.5 = 6075 veh
d. Determine the average individual delay. This is obtained by dividing the total delay by the number of vehicles affected by the incident = (6150/6075) = 1.010 hr

Varying Demand and Constant Service Rate

The procedure described in the previous section also can be used for varying demand and constant service rate, if it is assumed that the demand changes at specific times and not gradually increasing or decreasing. The analysis for a gradual increase or decrease is beyond the scope of this book. Interested readers may refer to any book on Traffic Flow Theory for additional information on this topic.

6.5.2 Stochastic Analyses of Queues

Using a stochastic approach to analyze queues considers the fact that certain traffic characteristics such as arrival rates are not always deterministic. In fact, arrivals at an intersection for example are deterministic or regular only when approach volumes are high. Arrival rates tend to be random for light to medium traffic. The stochastic approach is used to determine the probability that an arrival will be delayed, the expected waiting time for all arrivals, the expected waiting time of an arrival that waits, and so forth.

Several models have been developed that can be applied to traffic situations such as the merging of ramp traffic to freeway traffic, interactions at pedestrian crossings, and sudden reduction of capacity on freeways. This section will give only the elementary queuing theory relationships for a specific type of queue; that is, the single-channel queue. The theoretical development of these relationships is not included here. Interested readers may refer to any traffic flow theory book for a more detailed treatment of the topic.

A queue is formed when arrivals wait for a service or an opportunity, such as the arrival of an accepted gap in a main traffic stream, the collection of tolls at a tollbooth or of parking fees at a parking garage, and so forth. The service can be provided in a single channel or in several channels. Proper analysis of the effects of such a queue can be carried out only if the queue is fully specified. This requires that the following characteristics of the queue be given: (1) the characteristic distribution of arrivals, such as uniform, Poisson, and so on; (2) the method of service, such as first come–first served, random, and priority; (3) the characteristic of the queue length, that is, whether it is finite or infinite; (4) the distribution of service times; and (5) the channel layout, that is, whether there are single or multiple channels and, in the case of multiple channels, whether they are in series or parallel. Several methods for the classification of queues based on the above characteristics have been used—some of which are discussed below.

Arrival Distribution. The arrivals can be described as either a deterministic distribution or a random distribution. Light-to-medium traffic is usually described by a Poisson distribution, and this is generally used in queuing theories related to traffic flow.

Service Method. Queues also can be classified by the method used in servicing the arrivals. These include first come–first served where units are served in order of their arrivals, and last in-first served, where the service is reversed to the order of arrival. The service method can also be based on priority, where arrivals are directed to specific queues of appropriate priority levels—for example, giving priority to buses. Queues are then serviced in order of their priority level.

Characteristics of the Queue Length. The maximum length of the queue, that is, the maximum number of units in the queue, is specified, in which case the queue is a finite or truncated queue, or else there may be no restriction on the length of the queue. Finite queues are sometimes necessary when the waiting area is limited.

Service Distribution. The Poisson and negative exponential distributions have been used as the random distributions.

Number of Channels. The number of channels usually corresponds to the number of waiting lines and is therefore used to classify queues, for example, as a single-channel or multi-channel queue.

Oversaturated and Undersaturated Queues. Oversaturated queues are those in which the arrival rate is greater than the service rate, and undersaturated queues are those in which the arrival rate is less than the service rate. The length of an undersaturated queue may vary but will reach a steady state with the arrival of units. The length of an oversaturated queue, however, will never reach a steady state but will continue to increase with the arrival of units.

Single-Channel, Undersaturated, Infinite Queues

Figure 6.16 is a schematic of a single-channel queue in which the rate of arrival is q veh/h and the service rate is Q veh/h. For an undersaturated queue, $Q > q$, assuming that both the rate of arrivals and the rate of service are random, the following relationships can be developed:

1. Probability of n units in the system, $P(n)$:

$$P(n) = \left(\frac{q}{Q}\right)^n \left(1 - \frac{q}{Q}\right) \tag{6.62}$$

where n is the number of units in the system, including the unit being serviced.

2. The expected number of units in the system, $E(n)$:

$$E(n) = \frac{q}{Q - q} \tag{6.63}$$

3. The expected number of units waiting to be served (that is, the mean queue length) in the system, $E(m)$:

$$E(m) = \frac{q^2}{Q(Q - q)} \tag{6.64}$$

Note that E(m) is not exactly equal to E(n) − 1, the reason being that there is a definite probability of zero units being in the system, P (0).

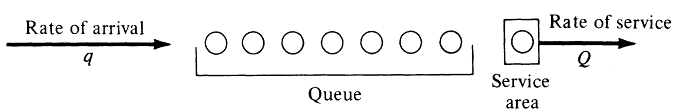


Figure 6.16 A Single-Channel Queue

4. Average waiting time in the queue, $E(w)$:

$$E(w) = \frac{q}{Q(Q - q)} \tag{6.65}$$

5. Average waiting time of an arrival, including queue and service, $E(v)$:

$$E(v) = \frac{1}{Q - q} \tag{6.66}$$

6. Probability of spending time t or less in the system:

$$P(v \leq t) = 1 - e^{-(1-\frac{q}{Q})qt} \tag{6.67}$$

7. Probability of waiting for time t or less in the queue:

$$P(w \leq t) = 1 - \frac{q}{Q}e^{-(1-\frac{q}{Q})qt} \tag{6.68}$$

8. Probability of more than N vehicles being in the system, that is, $P(n > N)$:

$$P(n > N) = \left(\frac{q}{Q}\right)^{N+1} \tag{6.69}$$

Equation 6.63 can be used to produce a graph of the relationship between the expected number of units in the system, $E(n)$, and the ratio of the rate of arrival to the rate of service, $\rho = q/Q$. Figure 6.17 is such a representation for different values of ρ. It should be noted that as this ratio tends to 1 (that is, approaching saturation), the expected number of vehicles in the system tends to infinity. This shows that q/Q, which

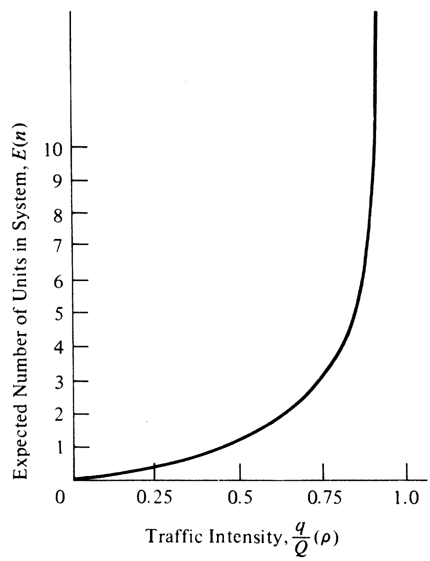


Figure 6.17 Expected Number of Vehicles in the System $E(n)$ versus Traffic Intensity (ρ)

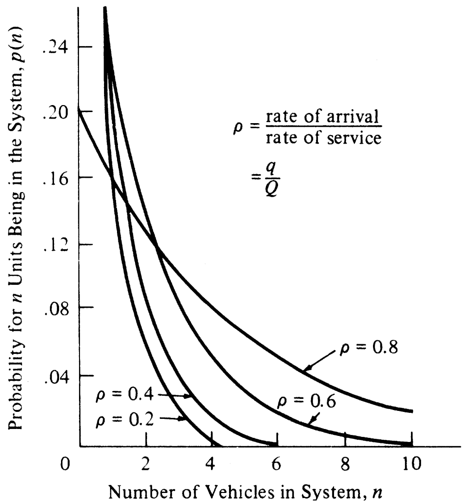


Figure 6.18 Probability of n Vehicles Being in the System for Different Traffic Intensities (ρ)

is usually referred to as the *traffic intensity,* is an important factor in the queuing process. The figure also indicates that queuing is of no significance when ρ is less than 0.5, but at values of 0.75 and above, the average queue lengths tend to increase rapidly. Figure 6.18 is also a graph of the probability of n units being in the system versus q/Q.

Example 6.10 Application of the Single-Channel, Undersaturated, Infinite Queue Theory to a Tollbooth Operation

On a given day, 425 veh/h arrive at a tollbooth located at the end of an off-ramp of a rural expressway. If the vehicles can be serviced by only a single channel at the service rate of 625 veh/h, determine (a) the percentage of time the operator of the tollbooth will be free, (b) the average number of vehicles in the system, and (c) the average waiting time for the vehicles that wait. (Assume Poisson arrival and negative exponential service time.)

Solution:

a. $q = 425$ and $Q = 625$. For the operator to be free, the number of vehicles in the system must be zero. From Eq. 6.62,

$$P(0) = 1 - \frac{q}{Q} = 1 - \frac{425}{625}$$
$$= 0.32$$

The operator will be free 32% of the time.

b. From Eq. 6.63,

$$E(n) = \frac{425}{625 - 425}$$
$$= 2$$

c. From Eq. 6.66,

$$E(v) = \frac{1}{625 - 425} = 0.005 \text{ hr}$$
$$= 18.0 \text{ sec}$$

Single-Channel, Undersaturated, Finite Queues

In the case of a finite queue, the maximum number of units in the system is specified. Let this number be N. Let the rate of arrival be q and the service rate be Q. If it is also assumed that both the rate of arrival and the rate of service are random, the following relationships can be developed for the finite queue.

1. Probability of n units in the system:

$$P(n) = \frac{1 - \rho}{1 - \rho^{N+1}} \rho^n \tag{6.70}$$

where $\rho = q/Q$.

2. The expected number of units in the system:

$$E(n) = \frac{\rho}{1 - \rho} \frac{1 - (N + 1)\rho^N + N\rho^{N+1}}{1 - \rho^{N+1}} \tag{6.71}$$

Example 6.11 Application of the Single-Channel, Undersaturated, Finite Queue Theory to an Expressway Ramp

The number of vehicles that can enter the on ramp of an expressway is controlled by a metering system which allows a maximum of 10 vehicles to be on the ramp at any one time. If the vehicles can enter the expressway at a rate of 500 veh/h and the rate of arrival of vehicles at the on ramp is 400 veh/h during the peak hour, determine (a) the probability of 5 cars being on the on ramp, (b) the percent of time the ramp is full, and (c) the expected number of vehicles on the ramp during the peak hour.

Solution:

a. Probability of 5 cars being on the on ramp: $q = 400$, $Q = 500$, and $\rho = (400/500) = 0.8$. From Eq. 6.70,

$$P(5) = \frac{(1 - 0.8)}{1 - (0.8)^{11}} (0.8)^5$$
$$= 0.072$$

b. From Eq. 6.70, the probability of 10 cars being on the ramp is

$$P(10) = \frac{1 - 0.8}{1 - (0.8)^{11}}(0.8)^{10}$$
$$= 0.023$$

That is, the ramp is full only 2.3% of the time.

c. The expected number of vehicles on the ramp is obtained from Eq. 6.71:

$$E(n) = \frac{0.8}{1 - 0.8}\,\frac{1 - (11)(0.8)^{10} + 10(0.8)^{11}}{1 - (0.8)^{11}} = 2.97$$

The expected number of vehicles on the ramp is 3.

6.6 SUMMARY

One of the most important current functions of a traffic engineer is to implement traffic control measures that will facilitate the efficient use of existing highway facilities, since extensive highway construction is no longer taking place at the rate it once was. Efficient use of any highway system entails the flow of the maximum volume of traffic without causing excessive delay to the traffic and inconvenience to the motorist. It is therefore essential that the traffic engineer understands the basic characteristics of the elements of a traffic stream, since these characteristics play an important role in the success or failure of any traffic engineering action to achieve an efficient use of the existing highway system.

This chapter has furnished the fundamental theories that are used to determine the effect of these characteristics. The definitions of the different elements have been presented, together with mathematical relationships of these elements. These relationships are given in the form of macroscopic models, which consider the traffic stream as a whole, and microscopic models, which deal with individual vehicles in the traffic stream. Using the appropriate model for a traffic flow will facilitate the computation of any change in one or more elements due to a change in another element. An introduction to queuing theory is also presented to provide the reader with simple equations that can be used to determine delay and queue lengths in simple traffic queuing systems.

PROBLEMS

6-1 Observers stationed at two sections XX and YY, 500 ft apart on a highway, record the arrival times of four vehicles as shown in the accompanying table. If the total time of observation at XX was 15 sec, determine (a) the time mean speed, (b) the space mean speed, and (c) the flow at section XX.

Time of Arrival

Vehicle	*Section XX*	*Section YY*
A	T_0	$T_0 + 7.58$ sec
B	$T_0 + 3$ sec	$T_0 + 9.18$ sec
C	$T_0 + 6$ sec	$T_0 + 12.36$ sec
D	$T_0 + 12$ sec	$T_0 + 21.74$ sec

6-2 Data obtained from aerial photography showed six vehicles on a 600 ft-long section of road. Traffic data collected at the same time indicated an average time headway of 4 sec. Determine (a) the density on the highway, (b) the flow on the road, and (c) the space mean speed.

6-3 Two sets of students are collecting traffic data at two sections, *xx* and *yy,* of a highway 1500 ft apart. Observations at *xx* show that five vehicles passed that section at intervals of 3, 4, 3, and 5 sec, respectively. If the speeds of the vehicles were 50, 45, 40, 35, and 30 mi/h, respectively, draw a schematic showing the locations of the vehicles 20 sec after the first vehicle passed section *xx*. Also determine (a) the time mean speed, (b) the space mean speed, and (c) the density on the highway.

6-4 Determine the space mean speed for the data given in Problem 6.3 using the Garber and Sankar expression given in Eq. 6.5. Compare your answer with that obtained in Problem 6.3 for the space mean speed and discuss the results.

6-5 The data shown below were obtained on a highway. Use regression analysis to fit these data to the Greenshields model and determine (a) the mean free speed, (b) the jam density, (c) the capacity, and (d) the speed at maximum flow.

Speed (mi/h)	*Density (veh/mi)*
14.2	85
24.1	70
30.3	55
40.1	41
50.6	20
55.0	15

6-6 Under what traffic conditions will you be able to use the Greenshields model but not the Greenberg model? Give the reason for your answer.

6-7 The table below shows data on speeds and corresponding densities on a section of a rural collector road. If it can be assumed that the traffic flow characteristics can be described by the Greenberg model, develop an appropriate relationship between the flow and density. Also determine the capacity of this section of the road.

Speed (mi/h)	*Density (veh/mi)*	*Speed (mi/h)*	*Density (veh/mi)*
60.0	20	32.6	50
46.0	32	30.8	53
40.8	38	28.4	57
39.3	40	24.7	65
35.7	45	18.5	80

6-8 Researchers have used analogies between the flow of fluids and the movement of vehicular traffic to develop mathematical algorithms describing the relationship among traffic flow elements. Discuss in one or two paragraphs the main deficiencies in this approach.

6-9 Assuming that the expression:

$$\bar{u}_s = u_f e^{-k/k_j}$$

can be used to describe the speed-density relationship of a highway, determine the capacity of the highway from the data below using regression analysis.

k (veh/mi)	$\bar{u}_s$ *(mi/h)*
43	38.4
50	33.8
8	53.2
31	42.3

Under what flow conditions is the above model valid?

6-10 Results of traffic flow studies on a highway indicate that the flow-density relationship can be described by the expression:

$$q = u_f k - \frac{u_f}{k_j} k^2$$

If speed and density observations give the data shown below, develop an appropriate expression for speed versus density for this highway, and determine the density at which the maximum volume will occur as well as the value of the maximum volume. Also plot speed versus density and volume versus speed for both the expression developed and the data shown. Comment on the differences between the two sets of curves.

Speed (mi/h)	*Density (veh/mi)*
50	18
45	25
40	41
34	58
22	71
13	88
12	99

6-11 Traffic on the eastbound approach of a signalized intersection is traveling at 40 mi/h, with a density of 44 veh/mi/ln. The duration of the red signal indication for this approach is 30 sec. If the saturation flow is 19500 veh/h/ln with a density of 51 veh/mi/ln, and the jam density is 120 veh/mi/ln, determine the following:

(a) The length of the queue at the end of the red phase
(b) The maximum queue length
(c) The time it takes for the queue to dissipate after the end of the red indication.

6-12 A developer wants to provide access to a new building from a driveway placed 1000 ft upstream of a busy intersection. He is concerned that queues developing during the red phase of the signal at the intersection will block access. If the speed on the approach averages 35 mi/h, the density is 50 veh/mi, and the red phase is 20 sec, determine if the driveway will be affected. Assume that the traffic flow has a jam density of 110 veh/mi and can be described by the Greenshields model.

6-13 Studies have shown that the traffic flow on a two-lane road adjacent to a school can be described by the Greenshields model. A length of 0.5 mi adjacent to a school is described as a school zone (see Figure 6.19) and operates for a period of 30 min just before the start of school and just after the close of school. The posted speed limit for the school zone during its operation is 20 mi/h. Data collected at the site when the school zone is not in operation show that the jam density and mean free speed for each lane are 118 veh/mi and 63 mi/h. If the demand flow on the highway at the times of operation of the school zone is 95% of the capacity of the highway, determine:

(a) The speeds of the shock waves created by the operation of the school zone
(b) The number of vehicles affected by the school zone operation
(c) The time the queue takes for it to dissipate after the operation of the school zone

6-14 Briefly describe the different shock waves that can be formed and the traffic conditions that will result in each of these shock waves.

6-15 Traffic flow on a four-lane (one direction) freeway can be described by the Greenshields model. Two lanes of the four lanes on a section of this freeway will have to be closed to undertake an emergency bridge repair that is expected to take 2 hr. The mean free speed of the highway is 60 mi/h and the jam density is 140 veh/mi/ln. If it is estimated that the demand flow on the highway during the emergency repairs is 80% of the capacity, using the deterministic approach, determine:

(a) The maximum queue length that will be formed
(b) The total delay
(c) The number of vehicles that will be affected by the incident
(d) The average individual delay

6-16 Repeat Problem 6-15 for the expected repair periods of 1 hr, 1.5 hr, 2.5 hr, 2.75 hr, and 3 hr. Plot a graph of average individual delay vs the repair period and use this graph to discuss the effect of the expected repair time on the average delay.

6-17 Repeat Problem 6-15 for the expected demand flows of 60%, 70%, 75%, and 85% of the capacity of the highway. Plot a graph of average individual delay vs the expected

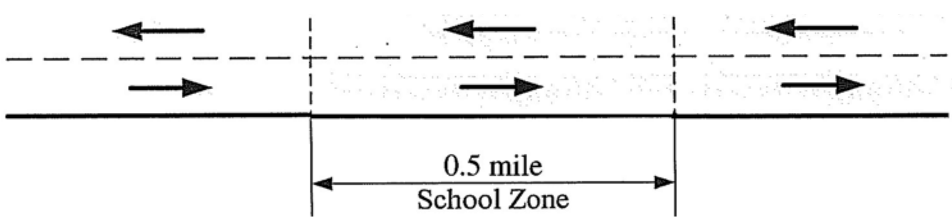


Figure 6.19 Layout of School Zone for Problem 6-13

demand flow and use this graph to discuss the effect of the expected demand flow on the average delay.

6-18 Traffic flow on a section of a two-lane highway can be described by the Greenshields model, with a mean free speed of 55 mi/h and a jam density of 145 veh/mi/ln. At the time when the flow was 90% of the capacity of the highway, a large dump truck loaded with heavy industrial machinery from an adjacent construction site joins the traffic stream and travels at a speed of 15 mi/h for a length of 3.5 mi along the upgrade before turning off onto a dump site. Due to the relatively high flow in the opposite direction, it is impossible for any car to pass the truck. Determine how many vehicles will be in the platoon behind the truck by the time the truck leaves the highway.

6-19 Briefly discuss the phenomenon of gap acceptance with respect to merging and weaving maneuvers in traffic streams.

6-20 The table below gives data on accepted and rejected gaps of vehicles on the minor road of an unsignalized intersection. If the arrival of major road vehicles can be described by the Poisson distribution, and the peak hour volume is 1100 veh/h, determine the expected number of accepted gaps that will be available for minor road vehicles during the peak hour.

Gap (t) (s)	*Number of Rejected Gaps > t*	*Number of Accepted Gaps > t*
1.5	92	3
2.5	52	18
3.5	30	35
4.5	10	62
5.5	2	100

6-21 Using appropriate diagrams, describe the resultant effect of a sudden reduction of the capacity (bottleneck) on a highway both upstream and downstream of the bottleneck.

6-22 The capacity of a highway is suddenly reduced to 60% of its normal capacity due to closure of certain lanes in a work zone. If the Greenshields model describes the relationship between speed and density on the highway, the jam density of the highway is 112 veh/mi, and the mean free speed is 64.5 mi/h, determine by what percentage the space mean speed at the vicinity of the work zone will be reduced if the flow upstream is 80% of the capacity of the highway.

6-23 The arrival times of vehicles at the ticket gate of a sports stadium may be assumed to be Poisson with a mean of 30 veh/h. It takes an average of 1.5 min for the necessary tickets to be bought for occupants of each car.

(a) What is the expected length of queue at the ticket gate, not including the vehicle being served?
(b) What is the probability that there are no more than 5 cars at the gate, including the vehicle being served?
(c) What will be the average waiting time of a vehicle?

6-24 An expressway off-ramp consisting of a single lane leads directly to a tollbooth. The rate of arrival of vehicles at the expressway can be considered to be Poisson with a mean of 50 veh/h, and the rate of service to vehicles can be assumed to be exponentially distributed with a mean of 1 min.

(a) What is the average number of vehicles waiting to be served at the booth (that is, the number of vehicles in queue, not including the vehicle being served)?
(b) What is the length of the ramp required to provide storage for all exiting vehicles 85% of the time? Assume the average length of a vehicle is 20 ft and that there is an average space of 5 ft between consecutive vehicles waiting to be served.
(c) What is the average waiting time a driver waits before being served at the tollbooth (that is, the average waiting time in the queue)?